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THE
METHOD
OF
INCREMENTS.

WHEREIN
The PRINCIPLES are demonstrated;
AND
The PRACTICE thereof shewn in the Solution of
PROBLEMS.

Nil tam difficile est, quod non Solertia vincat.

LIL. GRAM.

by Emerson

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PROBLEMS

By the Author of the
ART OF LOGIC
AND
THE ART OF THINKING
Corrected Edition

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T H E
P R E F A C E.

HE that writes upon a new or strange subject, little treated on before ; has something more to do, than he that writes only on common or known matters. For he not only has his general scheme to lay, and the several parts of his work to connect properly together ; but he has numberless computations to go through, which never before have been attempted ; and which, without trial, he is ignorant whether they will succeed or not ; and like a traveller wandering in an unfrequented and desolate country, must often turn back and begin his progress anew. But this is not the fate of a common writer, his road lies plain before him, and he cannot be at a Loss to find it, where so many people have led the way before ; and his labour is often no more than mere transcribing.

In the present affair, as I set off in an unfrequented path, where few have gone before me ; and as my subject lies out of the common road ; I have taken no little pains, in order to treat it with all imaginable perspicuity, to make the Principles intelligible, and the practice easy. And yet I have consulted brevity as much as I could, for fear of being thought tedious, or that I was writing only for the booksellers.

The

The Inventor of the Method of Increments was the learned Dr. Taylor, who, in the year 1717, published a treatise of that method; and afterwards gave some farther explanation of the same in the Philosophical Transactions, as applied to the finding the sums of series. But as the Doctor's writings upon this subject, are all in Latin, and have never been translated entire; and as his way of writing is so very short and abstracted, as not to be obvious to common readers; it has happened, that this elegant method of computation has lain ever since in obscurity, has been read by few, and improved by none.

The Differential Method of Mr. Sterling, which he applies to the summation and interpolation of series, is of the same nature as the Method of Increments, but not so general and extensive. Neither does he use the Doctor's way of notation, but proceeds in a method of his own, sufficient to answer the uses he applies it to. But both these authors frequently leave their proper subjects, and make digressions into Fluxions, as there every where appears to be a great analogy between the two methods.

In the following book, I have extracted from both these authors so much of this subject, as I thought necessary to compleat my plan, and have so methodized and combined them together, and made such farther additions thereto, as I thought was wanting for advancing both the theory and practice, and for reducing this art to a regular, uniform, and general method. Wherein I first lay down and demonstrate the Principles it depends on; and then I apply them to practice in the solution of problems. In which, I hope, I shall not confound my readers, by running from one subject to another, but keep close to my present business.

But

The P R E F A C E.

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But I ought to observe, that I use a way of notation something different from Dr. Taylor. He puts $\dot{z}$, $\ddot{z}$, $\dddot{z}$, &c. for the preceding values; and $\underset{1}{z}$, $\underset{2}{z}$, $\underset{3}{z}$, &c. for succeeding values of z . And he puts $\underset{1}{z}$, $\underset{2}{z}$, $\underset{3}{z}$, &c. for $\dot{z}$, $\ddot{z}$, $\dddot{z}$, &c. the number written underneath, signifying the number of points. But as the different values of z , far oftener occur to me, in the practice, than different numbers of points, and greater in degree too; I find it more convenient to express the different values of z by numbers, viz. preceding values by negative numbers, and succeeding values by affirmative ones, written underneath. And for the increment of any order, I put its proper number of points, as that number is never very great.

By help of this method, a great many beautiful problems are easily solved, which scarce admit of a solution any other way. As, suppose several series of quantities be given, whose terms are all formed according to some certain law, which you have given; the Method of Increments will find out a general Series, which comprehends all particular cases, and from which all of that kind may be found. And though such rules may sometimes be gained by induction, that is, by trying many particular cases; yet this is neither so scientific, nor so satisfactory to the mind, as the other. For there will still remain some doubt, whether the cases we have not yet tried, be conformable to this rule. But such a rule, gained by the Method of Increments, is general, and puts it out of all dispute.

The Method of Increments is also of great use in finding any Term of a series proposed; for the law being given by which the terms are
a
formed;

formed; by help of this general law, the Method of Increments will help us to this term, either expressed in finite quantities, or by an infinite series.

Another use of the Method of Increments is to find the sums of series; this method will find the sums of great numbers of them in finite terms. And when the sum of a series cannot be had in finite terms, we must have recourse to infinite series; for the integral being expressed by such a series, the sum of a competent number of the terms thereof will give the sum of the series required. And this is equivalent to transforming one series into another converging quicker: and sometimes a very few terms of this series will give the sum of the series required. Not that all sorts of series can be summed up this way; for there are many of them, whose sums cannot be had by any rules yet discovered.

The last thing I shall mention, of the use of this method; is this, that from hence the principal foundation of the Method of Fluxions, may be easily derived. For as in the Method of Increments, the increment may be of any magnitude, so in the Method of Fluxions, it must be supposed infinitely small; whence all preceding and successive values of the variable quantity will be equal, from which equality the rules for performing the principal operations of Fluxions are immediately deduced.

That I may give the reader a more perfect idea of the nature of this method: suppose the abscissa of a curve be divided into any number of equal parts, each part whereof is called the increment of the abscissa; and imagine so many parallelograms to be erected thereon; either circumscribing the curvilinear figure, or inscribed in it; then the
finding

finding the sum of all these parallelograms is the business of the Method of Increments. But if the parts of the abscissa be taken infinitely small, then these parallelograms degenerate into the curve; and then it is the business of the Method of Fluxions, to find the sum of all, or the area of the curve. So that the Method of Increments finds the sum of any number of finite quantities; and the Method of Fluxions, the sum of an infinite number of infinitely small ones: and this is the essential difference between these two methods.

There is such a near relation between the Method of Fluxions, and that of Increments, that many of the rules for the one, with little variation, serve also for the other. And in the course of this work, I have handled it so, as to make that relation appear, by making the correspondent operations similar in both of them. And here, as in the Method of Fluxions, some questions may be solved, and the integrals found, in finite terms; whilst in others we are forced to have recourse to infinite series for a solution. And the like difficulties will occur in the Method of Increments, as usually happen in Fluxions. For whilst some fluxionary quantities have no fluents, but what are expressed by series; so some Increments have no integrals, but what infinite series afford; which will often, as in Fluxions, diverge and become useless.

As I cannot promise, that I shall have time and leisure hereafter, to prosecute this subject any farther. And as I have an earnest desire for advancing truth and improving science; let me here intreat the friendly Mathematicians, who are lovers of science, to lend their kind assistance, for the advancement of this uncultivated branch of knowledge, yet in its infancy, or rather, as yet in the hands of Lucina;
either

either according to the model I have here laid before them, or some better, if it can be found; so that by degrees it may at length be brought to perfection: to whose generous endeavours I heartily wish good success.

W. Emerson.

THE

THE
M E T H O D
O F
I N C R E M E N T S.

DEFINITIONS.

DEFIN. I.

WHEN a quantity is supposed to increase (or decrease) by certain steps or degrees, it is called an *Integral*.

DEFIN. II.

The increase of any quantity from its present value, to the next succeeding value, is called an *Increment*: or, if it decreases, a *Decrement*.

DEFIN. III.

The increase of any increment, is its *second increment*; and the increase of the second increment is the *third increment*; and so on.

DEFIN. IV.

Succeeding values are the several values of the integral, succeeding one another in regular order, from the present value; and *preceding values*, are those that arise before the present value. All these are called *Factors*.

DEFIN. V.

A *perfect quantity*, is such as contains any number of successive values without intermission, and a *defective quantity*, is that where some successive value or values are wanting.

Thus $\overset{x}{\underset{2}{x}} \overset{x}{\underset{3}{x}} \overset{x}{\underset{4}{x}} \overset{x}{\underset{5}{x}}$ is a perfect quantity; and $\overset{x}{\underset{2}{x}} \overset{x}{\underset{4}{x}} \overset{x}{\underset{5}{x}}$, an imperfect, or defective one.

NOTATION.

1. ANY letters you please are put for simple integral quantities, as s, t, v, x, y, z , &c. or even a, b, c, d , &c. taking notice always to mention, what are constant, and what are variable quantities.

2. The different values of a simple integral are denoted by the same letter with small figures under them; thus if z be an integral, then $\underset{1}{z}, \underset{2}{z}, \underset{3}{z}, \underset{4}{z}$, &c. are the present value, and the first, second, third, &c. successive values thereof; and the preceeding values are denoted with figures and negative signs; thus $\underset{-1}{z}, \underset{-2}{z}, \underset{-3}{z}, \underset{-4}{z}$, are the first, second, third, fourth preceding values, and the figure denoting any value is the *characteristic*.

3. The increments are denoted with the same letters and points under them, thus $\overset{x}{x}$ is the increment of x , and $\overset{z}{z}$ is the increment of z . Also $\overset{x}{\underset{1}{x}}$ is the increment of $\overset{x}{\underset{1}{x}}$; and $\overset{x}{\underset{2}{x}}$ of $\overset{x}{\underset{2}{x}}$, &c.

4. The second, third, &c. increment is denoted with two, three, &c. points; as $\overset{\overset{x}{x}}{x}$ is the second increment of x , and $\overset{\overset{\overset{z}{z}}{z}}{z}$ is the third increment of z ; and so on. And these are denominated increments of such an order, according to the number of points.

5. If x be any increment, then $[x]$ denotes the integral of x , and $\overset{2}{[x]}$ denotes the integral of $[x]$, or the second integral of x : and $\overset{3}{[x]}$ is the third integral of x , or an integral of the third order, &c.

6. Quantities written thus, $\overset{x}{\underset{1}{x}} \cdots \overset{n}{\underset{5}{x}}$, signifies $\overset{x}{\underset{1}{x}} \overset{n}{\underset{2}{x}} \overset{n}{\underset{3}{x}} \overset{n}{\underset{4}{x}} \overset{x}{\underset{5}{x}}$, or that the quantities are continued from the first to the last.

SCHOLIUM.

HENCE any quantity, which is not a given one, may be considered either as an integral, or an increment.

PROP.

P R O P O S I T I O N I.

IF z be any integral increasing (or decreasing) uniformly, then I say, $z = z + n z$, where n may be affirmative, negative, or fractional.

This appears from the notation, for if $n=1$, then $z + z = z$; if $n=2$, $z + z$ or $z + 2z = z$; if $n=3$, $z + z$ or $z + 3z = z$, &c. therefore universally $z = z + n z$. Also $z = z + \frac{1}{2}z$, and $z = z + \frac{2}{3}z$, &c. and $z = z - z$, $z = z - 2z$, &c.

C O R O L. I.

HENCE it follows, that any value of the integral may be expressed an infinite variety of ways, as $x = x + x = x + 2x = x + 3x = x + 4x = x + 5x = x + 6x$, &c.

Likewise $x = x - x = x - 2x = x - 3x = x - 4x = x - 5x = x - 6x = x - 7x$, &c. Also $x + x = x$.

And in general $x = x + m - r . x$.

C O R O L. II.

HENCE also, if any integral be subtracted from its next succeeding value, the remainder is the increment of that integral. And therefore when any doubt arises about the increment and corresponding integral, this rule being applied will always determine it.

C O R O L. III.

WHEN any integral quantity decreases, its increment will be negative.

P R O P.

PROP. II.

IF z be an integral increasing (or decreasing) unequally; then I say

$$z = z + n\underset{n}{z} + \underset{2}{n \cdot n-1}\underset{2}{z} + \underset{2}{n \cdot n-1 \cdot n-2}\underset{3}{z} + \underset{2 \cdot 3 \cdot 4}{n \cdot n-1 \cdot n-2 \cdot n-3}\underset{4}{z}, \&c.$$
 And
 where n represents a negative number, make the even terms negative, still
 preserving n affirmative.

This will appear by the following table, where the first column gives the successive values of z , the second column the increments of those in the first; the third column the increments of the second; the fourth, the increments of the third, &c.

Succeeding values of z .	1 Differences.	2 Differences.	3 Differen.	4 Diff.
$z = z$	$\underset{n}{z}$	$\underset{n}{z}$	$\underset{n}{z}$	$\underset{n}{z}$
$z = z + \underset{n}{z}$	$\underset{n}{z} + \underset{n}{z}$	$\underset{n}{z} + \underset{n}{z}$	$\underset{n}{z} + \underset{n}{z}$	$\underset{n}{z}$
$z = z + 2\underset{n}{z} + \underset{2}{z}$	$\underset{n}{z} + 2\underset{n}{z} + \underset{2}{z}$	$\underset{n}{z} + 2\underset{n}{z} + \underset{2}{z}$	$\underset{n}{z} + \underset{n}{z}$	$\&c.$
$z = z + 3\underset{n}{z} + 3\underset{2}{z} + \underset{3}{z}$	$\underset{n}{z} + 3\underset{n}{z} + 3\underset{2}{z} + \underset{3}{z}$	$\underset{n}{z} + 2\underset{n}{z} + \underset{2}{z}$	$\&c.$	$\&c.$
$z = z + 4\underset{n}{z} + 6\underset{2}{z} + 4\underset{3}{z} + \underset{4}{z}$	$\&c.$	$\&c.$		
$\&c.$				

The first column is continued at pleasure, by continually adding each value to its increment in the second column. And by these values in the first column thus obtained, it appears that the coefficients of z , $\underset{n}{z}$, $\underset{2}{z}$, &c. are the same as the coefficients of the corresponding power of a binomial, and therefore each in general is represented by $1 + n + \frac{n \cdot n-1}{2} + \frac{n \cdot n-1 \cdot n-2}{2 \cdot 3}$, &c. And when n is negative, the signs of the second, fourth, sixth, &c. terms in the values of $\underset{n}{z}$, $\underset{2}{z}$, $\underset{3}{z}$, &c. will be negative, which produces the same rule.

COROL. I.

THE first, second, third, &c. increment of a variable quantity is no more than the first term of the first, second, and third, &c. differences of that quantity, and its successive values; as appears by the table.

COROL.

INCREMENTS.

5

COROL. II.

AND therefore the m^{th} succeeding value of the n^{th} increment is the same as the n^{th} increment of the m^{th} succeeding value. For example, $z+z$ is the next succeeding value of z , and it is also the increment of $z+z$, or $\frac{z}{1}$; that is, $\frac{z}{1} = \frac{z}{1}$. Likewise $\frac{z}{2} = \frac{z}{2}$, and $\frac{z}{1} = \left[\frac{z}{1} \right] = \left[\frac{z}{1} \right] = \left[\frac{z}{1} \right] = \left[\frac{z}{1} \right]$, &c.

COROL. III.

IF z increases unequally; then the increment of $z \dots z = z \dots z \times \frac{1}{n}$

$$\frac{n+1}{1} \cdot z + \frac{n}{2} z A + \frac{n-1}{3} B z + \frac{n-2}{4} C z \&c.$$

For the increment $\frac{z \dots z}{n+1} - \frac{z \dots z}{n} = \frac{z \dots z}{n} \times \frac{z - z}{n+1} = \frac{z \dots z}{n} \times \frac{z - z}{n+1}$
 $\times \frac{z - z}{m} = \frac{z \dots z}{n} \times \frac{z + m z + \frac{m \cdot m-1}{2} z, \&c. - z}{n+1}$

COROL. IV.

ALSO, If z increases unequally; the increment of

$$\frac{1}{z \dots z} = \frac{\frac{n+1}{1} \times z + \frac{n}{2} A z + \frac{n-1}{3} B z + \frac{n-2}{4} C z, \&c.}{z \dots z}$$

$$\text{For } \frac{1}{z \dots z} - \frac{1}{z \dots z} = - \frac{\frac{n+1}{1} \cdot z + \frac{n}{2} A z + \frac{n-1}{3} B z + \frac{n-2}{4} C z, \&c. - z}{z \dots z}$$

SCHOLIUM.

IF $z, z, z, \&c.$ be $= 0$, then $z = z + n z$, which agrees with Prop. I.

C

PROP.

PROP. III.

I *F* z increases uniformly, then $\frac{1}{z} = \frac{1}{z} - \frac{m}{z} A z - \frac{m-1}{z} B z - \frac{m-2}{z} C z - \frac{m-3}{z} D z - \&c.$ Where $A, B, C, D, \&c.$ are the preceding terms with their signs.

$$\text{For } \frac{1}{z} = \frac{z \dots z}{m-1} = \frac{z \dots z \times \overline{z-mz}}{z \dots z} = \frac{1}{z} - \frac{mz \times z \dots z}{z \dots z},$$

$$\text{Also } \frac{z \dots z}{m-1} = \frac{z \dots z \times \overline{z-m-1z}}{z \dots z} = \frac{1}{z} - \frac{m-1z \times z \dots z}{z \dots z},$$

$$\text{Also } \frac{z \dots z}{m-1} = \frac{z \dots z \times \overline{z-m-2z}}{z \dots z} = \frac{1}{z} - \frac{m-2z \times z \dots z}{z \dots z},$$

$$\text{Also } \frac{z \dots z}{m-1} = \frac{z \dots z \times \overline{z-m-3z}}{z \dots z} = \frac{1}{z} - \frac{m-3z \times z \dots z}{z \dots z},$$

&c. Whence

$$\frac{1}{z} = \frac{1}{z} - \frac{m}{z} A z + \frac{m}{z} \times \frac{m-1}{z} B z - \frac{m}{z} \times \frac{m-1}{z} \times \frac{m-2}{z} C z + \frac{m}{z} \times \frac{m-1}{z} \times \frac{m-2}{z} \times \frac{m-3}{z} D z - \&c. \text{ that is,}$$

$$\frac{1}{z} = \frac{1}{z} - \frac{m}{z} A z - \frac{m-1}{z} B z - \frac{m-2}{z} C z - \frac{m-3}{z} D z - \frac{m-4}{z} E z \&c.$$

COROL. I.

HENCE

$$\frac{1}{z} = \frac{1}{z} - \frac{m-1}{z} A z - \frac{m-2}{z} B z - \frac{m-3}{z} C z - \&c.$$

$$\frac{1}{z} = \frac{1}{z} - \frac{m-2}{z} A z - \frac{m-3}{z} B z - \frac{m-4}{z} C z - \&c.$$

$$\frac{1}{z} = \frac{1}{z} - \frac{m-3}{z} A z - \frac{m-4}{z} B z - \frac{m-5}{z} C z - \&c.$$

And in general,

$$\frac{1}{z} = \frac{1}{z} - \frac{m-n}{z} A z - \frac{m-n-1}{z} B z - \frac{m-n-2}{z} C z - \&c.$$

C O R O L. II.

HENCE also

$$\frac{1}{z z} = \frac{1}{z z} - \frac{m-1}{z} A z - \frac{m-2}{z} B z \&c.$$

$$\frac{1}{z z z} = \frac{1}{z z z} - \frac{m-2}{z} A z - \frac{m-3}{z} B z \&c.$$

$$\frac{1}{z z z z} = \frac{1}{z z z z} - \frac{m-3}{z} A z - \frac{m-4}{z} B z \&c.$$

And $\frac{1}{z \dots z z} = \frac{1}{z \dots z} - \frac{m-n-1}{z} A z - \frac{m-n-2}{z} B z \&c.$

Also $\frac{1}{z \dots z} = \frac{1}{z \dots z} - 0.$

P R O P. IV.

I F z be an integral increasing (or decreasing) unequally; I say

$$\frac{1}{z} = \frac{1}{z} - \frac{z}{z z} - \frac{z+z}{z z} - \frac{z+2z+z}{z z} - \frac{z+3z+3z+z}{z z}, \&c. \text{ continued to}$$

$m+1$ terms: where the numeral coefficients are the same as in the several powers of a binomial.

For

For let $A, B, C, D, \&c.$ be the increments of $\frac{1}{z}, \frac{1}{z}, \frac{1}{z}, \frac{1}{z}, \&c.$

$$\text{then } A = \frac{1}{z} - \frac{1}{z} = \frac{z-z}{z \cdot z} = \frac{-z}{z \cdot z}. \text{ Whence } \frac{1}{z} = \frac{1}{z} + A = \frac{1}{z} - \frac{z}{z \cdot z}.$$

$$\text{Again, } B = \frac{1}{z} - \frac{1}{z} = \frac{z-z}{z \cdot z} = (\text{by Prop. II.}) \frac{z+z-z-2z-z}{z \cdot z} = -\frac{z+z}{z \cdot z}.$$

$$\text{And therefore, } \frac{1}{z} = \frac{1}{z} + B = \frac{1}{z} - \frac{z}{z \cdot z} - \frac{z+z}{z \cdot z}.$$

$$\text{Likewise, } C = \frac{1}{z} - \frac{1}{z} = \frac{z-z}{z \cdot z} = (\text{by Prop. II.})$$

$$\frac{z+2z+z-z-3z-3z-z}{z \cdot z} = -\frac{z+2z+z}{z \cdot z}.$$

$$\text{And } \frac{1}{z} = \frac{1}{z} + C = \frac{1}{z} - \frac{z}{z \cdot z} - \frac{z+z}{z \cdot z} - \frac{z+2z+z}{z \cdot z}.$$

$$\text{Again, } D = \frac{1}{z} - \frac{1}{z} = \frac{z-z}{z \cdot z} = (\text{by Prop. II.})$$

$$\frac{z+3z+3z+z-z-4z-6z-4z-z}{z \cdot z} = -\frac{z+3z+3z+z}{z \cdot z}.$$

$$\text{And therefore } \frac{1}{z} = \frac{1}{z} + D; \text{ and so forward.}$$

COROLLARY.

IF $z, z, \&c.$ be supposed $= 0$, then the series will give the value of $\frac{1}{z}$, where z increases uniformly. Whence Prop. III. may be derived.

PROP.

P R O P. V.

THE increment of a rectangle or product xv , is equal to the sum of the products of each quantity into the increment of the other + the product of their increments $= xv + vx + vx$.

For the increment of $xv = \overline{xv} - xv = \overline{x+x} \times \overline{v+v} - xv = xv + xv + vx + vx - xv = xv + vx + vx$.

C O R O L. I.

THE increment of a rectangle or product, is equal to either quantity $\times$ by the increment of the other, + the succeeding value of this other $\times$ increment of the former.

For $xv + vx + vx = \overline{x+x} \times v + vx = vx + \overline{xv} = xv + \overline{v+v} \times x = xv + vx$.

C O R O L. II.

THE n^{th} increment of $xv = \frac{xv}{(n)} + \frac{n \cdot \overline{xv}}{(n-1)} + \frac{n \cdot \overline{n-1}}{2} \times \frac{xv}{(n-2)} + \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3} \frac{xv}{(n-3)}, \&c.$ where (n) signifies n points, $(n-1)$ $n-1$ points, $(n-2)$ $n-2$ points, &c.

For by Cor. I. of this Proposition we have

$$1. \quad xv + \overline{xv} = \text{first increment of } xv.$$

And by the same Cor. the increment of that is,

$$2. \quad \overline{xv} + \overline{\overline{xv}} + \overline{\overline{\overline{xv}}} = \text{the second increment of } xv.$$

D

And

And by the same Corollary, the increment of the last is,

$$\begin{array}{r}
 x v + x v \\
 + 2 x v + 2 x v \\
 + x v + x v \\
 \hline
 3. \quad x v + 3 x v + 3 x v + x v, \text{ the third increment of } x v.
 \end{array}$$

And in the like manner the increment of the last is,

$$\begin{array}{r}
 x v + x v \\
 + 3 x v + 3 x v \\
 + 3 x v + 3 x v \\
 + x v + x v \\
 \hline
 4. \quad x v + 4 x v + 6 x v + 4 x v + x v, \text{ the fourth incr. of } x v.
 \end{array}$$

&c. where the coefficients are those of a binomial.

PROP. VI.

IF any quantity x increases uniformly, and a given quantity be multiplied into a perfect integral made up of x ; its increment is equal to that whole quantity, multiplied by the number of factors and the common increment, and divided by the least value of the variable quantity.

Let $a x x x$ be a perfect integral; then by Cor. 2. of Prop. I. its increment is

$$\begin{aligned}
 & a x x x - a x x x = a x x x - x = a x x x - x = a x x x - x + 5x - x - 2x \\
 & = a x x x \times 3x = \frac{a x x x}{x} \times 3x.
 \end{aligned}$$

Again, the increment of $a x x x$ is

$$\begin{aligned}
 & a x x x - a x x x = a x x x - x = a x x x - x + 4x = 3x \times a x x x = \\
 & 3a x x x
 \end{aligned}$$

In

In general,

$$\begin{aligned} \text{The increment of } x_{m+1} x_{m+2} \dots x_{m+n} &= x_{m+1} x_{m+2} \dots x_{m+n+1} - x_m x_{m+1} \dots x_{m+n} \\ &= \frac{x_{m+n+1} - x_m}{m+n+1 - m} \times x_{m+1} x_{m+2} \dots x_{m+n} = \frac{x_{m+n+1} - x_m}{n+1} \times x_{m+1} x_{m+2} \dots x_{m+n} \\ &= \frac{x_{m+n+1} - x_m}{n+1} \times x_{m+1} x_{m+2} \dots x_{m+n} \end{aligned}$$

C O R O L L A R Y.

HENCE the integral of a perfect increment, is equal to that increment multiplied by the next preceeding value of the variable quantity; and then dividing by the new number of factors, and the common increment.

$$\text{Thus the integral of } 3a x x x x \text{ is } \frac{3a x x x x}{3x} = a x x x.$$

P R O P. VII.

IF x increases uniformly, and a given quantity be divided by a perfect integral made of the successive values of x ; its increment is equal to the whole quantity multiplied by the number of factors, and by the constant increment with a negative sign, and the whole divided by the next succeeding value of the variable quantity.

For by Cor. 2. Prop. I. the increment of

$$\begin{aligned} \frac{a}{x x x} &= \frac{a}{x x x} - \frac{a}{x x x} = \frac{a x}{x \dots x} - \frac{a x}{x \dots x} = \frac{a}{x \dots x} \times \\ &= \frac{x+2x - x-5x}{x \dots x} = \frac{-3ax}{x \dots x} \end{aligned}$$

$$\begin{aligned} \text{Again, Increment of } \frac{1}{x x x} &= \frac{1}{x x x} - \frac{1}{x x x} = \frac{x - 4x - x + x}{x \dots x} \\ &= \frac{-3x}{x \dots x} \end{aligned}$$

In

In general

$$\begin{aligned} \text{Increment of } \frac{a}{x \dots x} &= \frac{a}{x \dots x} - \frac{a}{x \dots x} = \\ &= \frac{x - a x}{m \quad m+n+1} = \frac{x+m x - x - m+n+1 \cdot x}{x \dots x} a = \frac{-n+1 \cdot a x}{x \dots x} \end{aligned}$$

COROLLARY.

HENCE the integral of a perfect fractional increment, is equal to that increment multiplied by the greatest value of the variable quantity, and divided by the new number of factors, and the common increment with a negative sign.

$$\text{Thus the integral of } \frac{-3 a x}{x \dots x} \text{ is } \frac{-3 a x}{x \dots x \times -3x} = \frac{a}{x \dots x}$$

PROP. VIII.

$$\begin{aligned} \text{IF } x \text{ be any integral, the increment of any power of it, } x^n, \text{ is equal to} \\ n x^{n-1} x + \frac{n \cdot n-1}{2} x^{n-2} x^2 + \frac{n \cdot n-1 \cdot n-2}{2 \cdot 3} x^{n-3} x^3 + \\ \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4} x^{n-4} x^4 \&c. = x^n \times \frac{nx}{x} + \frac{n-1}{2x} A x + \\ \frac{n-2}{3x} B x + \frac{n-3}{4x} C x \&c. \end{aligned}$$

$$\begin{aligned} \text{For by Cor. 2. Prop. I. Increment of } x^n \text{ is } x^n - x^n = x^{n+1} - x^n = \\ x^n + n x^{n-1} x + \frac{n \cdot n-1}{2} x^{n-2} x^2 \&c. - x^n = n x^{n-1} x + \\ \frac{n \cdot n-1}{2} x^{n-2} x^2 + \frac{n \cdot n-1 \cdot n-2}{2 \cdot 3} x^{n-3} x^3 + \&c. \end{aligned}$$

COROL. I.

HENCE the index of the integral is 1 more than the index of the highest power of the increment.

COROL.

C O R O L. II.

THE increment of x^n is equal to $x^n \times : \frac{n \cdot x}{x} + \frac{n+1}{2x} A \cdot x +$
 $\frac{n+2}{3x} B \cdot x + \frac{n+3}{4x} C \cdot x \&c.$

For x^n or $\overline{x+x}^n = x^n \times : 1 + \frac{n \cdot x}{x+x} + \frac{n \cdot \overline{n+1}}{2} \times \frac{x^2}{x+x^2} +$
 $\frac{n \cdot \overline{n+1} \cdot \overline{n+2}}{2 \cdot 3} \times \frac{x^3}{x+x^3} \&c.$

P R O P. IX.

THE integral of XV, that is $[XV] = [X]V - [\overset{2}{X}]V +$
 $[\overset{3}{X}]V - [\overset{4}{X}]V + [\overset{5}{X}]V - \&c.$

For taking the increment of every term by Cor. I. Prop. V. we have
 (putting small letters instead of capitals for brevity's sake),

$$\begin{aligned} xv &= xv + [\overset{1}{x}]v \\ &\quad - [\overset{1}{x}]v - [\overset{2}{x}]v \\ &\quad \quad + [\overset{2}{x}]v + [\overset{3}{x}]v \\ &\quad \quad \quad - [\overset{3}{x}]v \&c. \end{aligned}$$

Here all the terms destroy one another except xv , therefore xv is the
 increment of the series; and the series is the integral of xv .

E

C O R O L.

COROL. I.

THE n^{th} integral of xv , that is,

$$\begin{aligned} \left[xv \right]_n &= \left[x \right]_n v - n \left[x \right]_{n+1} v + \frac{n \cdot n+1}{2} \times \left[x \right]_{n+2} v - \frac{n \cdot n+1 \cdot n+2}{2 \cdot 3} \times \\ &\left[x \right]_{n+3} v + \frac{n \cdot n+1 \cdot n+2 \cdot n+3}{2 \cdot 3 \cdot 4} \times \left[x \right]_{n+4} v - \&c. \end{aligned}$$

$$\text{For (1.) } \left[xv \right] = \left[x \right] v - \left[x \right]_1 v + \left[x \right]_2 v - \left[x \right]_3 v \&c.$$

by this Proposition, and by the same way finding the integral of every separate term, we have

$$\begin{aligned} \left[xv \right] &= \left[x \right] v - \left[x \right]_1 v + \left[x \right]_2 v - \left[x \right]_3 v \&c. \\ &\quad - \left[x \right]_1 v + \left[x \right]_2 v - \left[x \right]_3 v \\ &\quad + \left[x \right]_2 v - \left[x \right]_3 v \\ &\quad - \left[x \right]_3 v \text{ that is} \end{aligned}$$

$$(2.) \left[xv \right] = \left[x \right] v - 2 \left[x \right]_1 v + 3 \left[x \right]_2 v - 4 \left[x \right]_3 v \&c.$$

And finding the integral of every separate term of this last, the same way, we have

$$\begin{aligned} \left[xv \right] &= \left[x \right] v - \left[x \right]_1 v + \left[x \right]_2 v - \left[x \right]_3 v \\ &\quad - 2 \left[x \right]_1 v + 2 \left[x \right]_2 v - 2 \left[x \right]_3 v \\ &\quad + 3 \left[x \right]_2 v - 3 \left[x \right]_3 v \\ &\quad - 4 \left[x \right]_3 v \end{aligned}$$

that

that is

$$(3.) \left[\begin{smallmatrix} xv \\ 1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v - 3 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v + 6 \left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right] v - 10 \left[\begin{smallmatrix} x \\ 3 \end{smallmatrix} \right] v \&c.$$

And after the same Manner,

$$(4.) \left[\begin{smallmatrix} xv \\ 2 \end{smallmatrix} \right] = \left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right] v - 4 \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v + 10 \left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right] v - 20 \left[\begin{smallmatrix} x \\ 3 \end{smallmatrix} \right] v \&c.$$

But the series $1, n, \frac{n \cdot n+1}{2}, \frac{n \cdot n+1 \cdot n+2}{2 \cdot 3}, \&c.$ expresses all these coefficients.

C O R O L. II.

HENCE if two of these three quantities $\left[\begin{smallmatrix} xv \\ 1 \end{smallmatrix} \right]$, $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v$, and $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v$, be given; the third may be found: or if three of these four $\left[\begin{smallmatrix} xv \\ 1 \end{smallmatrix} \right]$, $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v$, $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right] v$, and $\left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right] v$, be given; the fourth may be found, and so on.

C O R O L. III.

IF $v = 0$, and $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right]$ be given, $\left[\begin{smallmatrix} xv \\ 1 \end{smallmatrix} \right]$ will be known; and if $v = 0$, and $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right]$, and $\left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right]$, be given; $\left[\begin{smallmatrix} xv \\ 2 \end{smallmatrix} \right]$ will be known.

And if $v = 0$, and also $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix} \right]$, $\left[\begin{smallmatrix} x \\ 2 \end{smallmatrix} \right]$, and $\left[\begin{smallmatrix} x \\ 3 \end{smallmatrix} \right]$ be given; $\left[\begin{smallmatrix} xv \\ 3 \end{smallmatrix} \right]$ will be known.

P R O P. X.

IF two integrals be equal, they will always be equal for all correspondent preceeding values, or succeeding values, of the variable quantities therein.

For

For such an equation being expressed by general quantities, must hold good in all cases. As if $z = b \overset{2}{x} \overset{3}{x} \overset{4}{x} + a$, then $z = b \overset{1}{x} \overset{3}{x} \overset{4}{x} + a$, $z = b \overset{2}{x} \overset{4}{x} \overset{5}{x} + a$, $z = b \overset{1}{x} \overset{2}{x} \overset{3}{x} + a$, &c. For since $z = b \overset{2}{x} \overset{3}{x} \overset{4}{x} + a$, therefore the increments are equal, or $z = 3 b \overset{3}{x} \overset{4}{x}$, and adding them, $z + z$ or $z = b \overset{2}{x} \overset{3}{x} \overset{4}{x} \overset{5}{x} + 3 \overset{3}{x} + a = b \overset{3}{x} \overset{4}{x} \overset{5}{x} + a$. Again, the increment of this last quantity is $z = 3 b \overset{3}{x} \overset{4}{x} \overset{5}{x}$, and by addition $z + z$ or $z = b \overset{2}{x} \overset{3}{x} \overset{4}{x} \overset{5}{x} \overset{6}{x} + 3 \overset{4}{x} + a = b \overset{3}{x} \overset{4}{x} \overset{5}{x} \overset{6}{x} + a$; where x increases uniformly, and a is constant; and so for any other values of z .

P R O P. XI.

TO find the increment of any variable quantity, or integral.

R U L E S.

1. If the quantity proposed be not fractional, and be a perfect integral consisting of the successive values of the variable quantity which increases uniformly: multiply the proposed integral by the number of factors, and change the least factor for an increment. By Prop. VI.

2. In fractional quantities, where the denominator is perfect, and the variable quantity increases uniformly: multiply the proposed integral by the number of factors, and by the constant increment with a negative sign, and take the next succeeding value into the denominator. By Prop. VII.

3. The increment of any power as x^n is found by this series, $n x^{n-1} x + \frac{n \cdot n-1}{2} x^{n-2} x^2 + \frac{n \cdot n-1 \cdot n-2}{2 \cdot 3} x^{n-3} x^3 + \&c.$ by

Prop. VIII. Or this, $x^n \times : \frac{nx}{x} + \frac{n \cdot n+1}{2} \times \frac{x^2}{x^2} + \frac{n \cdot n+1 \cdot n+2}{2 \cdot 3} \times \frac{x^3}{x^3},$ &c.

4. In compound quantities, subtract the given integral from the next succeeding value, and reduce the remainder, for the increment. By Cor. 2. Prop. I.

5. Sometimes the quantity requires to be transformed by the following Proposition, and then the increment may be found by the foregoing rules.

E X A M P L E

EXAMPLE I.

To find the increment of $a - 2x + 5v$, this will be $-2x + 5v$, where a is constant.

EXAMPLE II.

Let $b x x x x$ be proposed; its increment is $4 b x x x x$.

EXAMPLE III.

The increment of $a x x x$ is $3 a x x x$.

EXAMPLE IV.

Let $x \dots x$ be given, its increment is $m+n+1 \cdot x x$

EXAMPLE V.

The increment of $\frac{a}{x \dots x}$ is $\frac{-5 a x}{x \dots x}$.

EXAMPLE VI.

If $\frac{b}{x \dots x}$ be given, its increment is $\frac{-9 b x}{x \dots x}$.

EXAMPLE VII.

To find the increment of $a z z z z - 3 b z z z + \frac{c}{z z z z}$. It will be $4 a z z z z - 9 b z z z - \frac{4 c z}{z \dots z}$.

EXAMPLE VIII.

The increment of z^4 is $4z^3z + 6z^2z^2 + 4z z^3 + z^4$.

EXAMPLE IX.

If $\frac{1}{z^3}$ be proposed, its increment is $-\frac{3z}{z^4} + \frac{6z^2}{z^5} - \frac{10z^3}{z^6} + \frac{15z^4}{z^7}$, &c.

EXAMPLE X.

To find the increment of a^x , a being constant. The increment is $= a^x - a^x = a^{x+x} - a^x = a^x - 1 \times a^x$.

EXAMPLE XI.

The increment of $\frac{1}{a^x}$ is required, its increment is $= \frac{1}{a^x} - \frac{1}{a^x} = \frac{a^x - a^{x+x}}{a^x a^x} = \frac{1 - a^x}{a^x a^x} a^x = \frac{1 - a^x}{a^{x+x}}$.

EXAMPLE XII.

The increment of $\frac{x}{z}$ is $= \frac{x}{z} - \frac{x}{z} = \frac{zx - xz}{zz} = \frac{zx - xz}{zz}$.

EXAMPLE XIII.

The increment of $\frac{1}{x \dots x z \dots z} = \frac{1}{x^{m+1} z^{n+1}} - \frac{1}{x^m z^n}$
 $= \frac{xz - xz}{x^{m+1} z^{n+1}} = \frac{xz - x + m + 1 \cdot xz}{x^{m+1} z^{n+1}} = \frac{xz - x + m + 1 \cdot xz + n + 1 \cdot z}{x^{m+1} z^{n+1}}$
 $= - \frac{n + 1 \cdot xz + z \times m + 1 \cdot x}{x^{m+1} z^{n+1}}; x, z \text{ being constant.}$

EXAMPLE

E X A M P L E X I V .

The increment of $xy = \overline{xy} - xy = \overline{x+x} \times \overline{y+y} - xy = xy + y + \overline{y} \times x = xy + yx.$

E X A M P L E X V .

For the increment of x^2y^3 , here the increment is $\overline{x+x^2} \times \overline{y+y^3}$
 $= \overline{x^2y^3} = \overline{xxx + 2xx + x} \times \overline{yyy + 3y^2y + 3yyy + y^3} - x^2y^3$
 $= 3x^2yy + 3x^2yyy + x^2y + 2xy^3x + 6xy^2xy + 6xyxxy$
 $+ 2xxx^3 + y^3xxx + 3y^2yxx + 3yyyxx + y^3xxx.$

E X A M P L E X V I .

Given $ax^2 + bv^3 - cx^2v$, its increment is, $a \times \overline{x+x^2} + b \times \overline{v+v^3} - c \times \overline{x+x^2} \times \overline{v+v^3} - ax^2 - bv^3 + cx^2v = 2axx + axx + 3bv^2v + 3bv^2v + bv^3 - 2cvxx - cvxx - cx^2v - 2cx^2v - cvxx.$

E X A M P L E X V I I .

The increment of $x \dots x \times z \dots z$ is $\overline{x \dots x} \times \overline{z \dots z} - x \dots x \times z \dots z$
 $= \overline{x \dots x} \times \overline{z \dots z} - x \dots x \times z \dots z = x + m+1 \cdot x \times z + n+1 \cdot z - xz$
 $\times \text{ into } x \dots x \times z \dots z = n+1 \cdot xz + m+1 \cdot x \times z + n+1 \cdot z \times$
 $\text{ into } x \dots x \times z \dots z = n+1 \cdot xz + m+1 \cdot x \times z \times \text{ into } x \dots x \times z \dots z.$
 Where x, z are constant.

E X A M P L E

EXAMPLE XVIII.

To find the increment of the Log. z .

The increment of $\log. z = \log. z - \log. z = \log. \frac{z+z}{z} - \log. z = \log. \frac{z+z}{z}$ (by the properties of logarithms) $= \frac{z}{z} -$

$$\frac{z}{2z^2} + \frac{z^3}{3z^3} - \frac{z^4}{4z^4} - \&c.$$

Therefore increment hyp. $z = \frac{z}{z} - \frac{z}{2z^2} + \frac{z^3}{3z^3} - \frac{z^4}{4z^4} + \frac{z^5}{5z^5} \&c.$ And in common log. multiply the series by $M = .4342945$.

Otherwise,

Put $v = 2z + z$, then $\text{incr. log. } z = \log. \frac{z+z}{z} = \log. \frac{v+z}{v-z} =$ (by logarithms) $2M \times : \frac{z}{v} + \frac{z^3}{3v^3} + \frac{z^5}{5v^5} + \frac{z^7}{7v^7} \&c. = \text{incr. log. } z.$

EXAMPLE XIX.

To find the increment $\overline{\log. z^2}$.

Put $Z = \text{hyp. log. } z$, then the $\text{incr. } ZZ = \overline{Z+Z^2} - ZZ = 2ZZ + Z \cdot Z = 2 \overline{Z+Z} \times Z$; and Z is had by Examp. 14.

EXAMPLE XX.

To find the incr. $z \log. z$.

Let $Z = \log. z$, then $\text{incr. } zZ = Zz + zZ + zZ =$ (by Ex. 14.)

$$Zz + z - \frac{z^2}{2z} + \frac{z^3}{3zz} - \frac{z^4}{4z^3} \&c. + \frac{z^2}{z} - \frac{z^3}{2zz} + \frac{z^4}{3z^2}, \&c.$$

that is,

$$\text{Incr. } z \log. z = Zz + z + \frac{z^2}{1.2z} - \frac{z^3}{2.3zz} + \frac{z^4}{3.4z^3} - \frac{z^5}{4.5z^4} \&c.$$

EXAMPLE

E X A M P L E X X I.

The incr. y log. z is required.

Let $Z = \log. z$, then $\text{incr. } y Z = Z y + y Z$, that is,

$$\text{Incr. } y \log. z = y \log. z + y : \times \frac{z}{z} - \frac{z z}{2 z z} + \frac{z^3}{3 z^3}, \&c. \text{ by Ex. 14.}$$

E X A M P L E X X I I.

To find the increment of the number whose log. is v.

Here num. of v — num. of $v = n. \overline{v+v} - n. v =$ (by the nature

of logarithms) $n. v \times : 1 + m v + \frac{m^2 v}{2} + \frac{m^3 v}{2.3}, \&c. - n. v,$

that is $\text{incr. numb. of } v = n. v \times : m v + \frac{m^2 v}{2} + \frac{m^3 v}{2.3} + \frac{m^4 v}{2.3.4}$
 &c. where $m = 2.302585$ for the common logarithms, or $= 1$,
 for the hyp. logarithms.

E X A M P L E X X I I I.

To find the increment of $a x^n + b x^{n-1} + c x^{n-2} + d x^{n-3} + e x^{n-4}, \&c.$

Inc. $a x^n$	$a \times : n x^{n-1} x + \frac{n. n-1}{2} x^{n-2} x^2 + \frac{n. n-1. n-2}{6} x^{n-3} x^3 + \frac{n. n-1. n-2. n-3}{24} x^{n-4} x^4$
Inc. $b x^{n-1}$	$b \times \quad \frac{n-1. n-2}{2} x^{n-2} x + \frac{n-1. n-2. n-3}{6} x^{n-3} x^2 + \frac{n-1. . n-3. n-4}{6} x^{n-4} x^3$
Inc. $c x^{n-2}$	$c \times \quad \frac{n-2. n-3}{2} x^{n-3} x + \frac{n-2. n-3. n-4}{2} x^{n-4} x^2$
Inc. $d x^{n-3}$	$d \times \quad n-3. x^{n-4} x$

G

Therefore

Therefore the increment of $a x^n + b x^{n-1} + c x^{n-2}$, &c. is

$$\begin{aligned}
 &= n a x^{n-1} x + \frac{n a}{2} x^2 \left\{ \begin{array}{l} \frac{n-1}{n-1} x^{n-2} + \frac{n \cdot n-1}{6} a x^3 \\ + b x \end{array} \right\} \left\{ \begin{array}{l} + \frac{n \cdot n-1 \cdot n-2}{24} a x^4 \\ + \frac{n-1 \cdot n-2}{6} b x^3 \\ + \frac{n-2}{2} c x^2 \\ + d x \end{array} \right\} \left\{ \begin{array}{l} \frac{n-3}{n-3} x^{n-3} \\ \times x^{n-3} \end{array} \right\} \\
 &+ \frac{n \cdot \dots \cdot n-3}{120} a x^5 \left\{ \begin{array}{l} \frac{n-1 \cdot \dots \cdot n-3}{24} b x^4 \\ + \frac{n-2 \cdot \dots \cdot n-3}{6} c x^3 \\ + \frac{n-3}{2} d x^2 \\ + e x \end{array} \right\} \left\{ \begin{array}{l} \frac{n-4}{n-4} x^{n-4}, \text{ \&c.} \end{array} \right\}
 \end{aligned}$$

SCHOLIUM.

FROM hence may be demonstrated the fundamental principles of *Fluxions*. For instance,

1. By Ex. 2. we have the incr. of $b x \dots x = 4 b x \times x x x$. Now let the increment x be infinitely small, and it changes into the fluxion $\dot{x}$, and all the values of $x, x, x, \&c.$ become equal to x , that is, $b x \dots x = b x^4$, and $4 b x x x x = 4 b \dot{x} x^3$. Therefore, taking the fluxion instead of the increment, the fluxion of $b x^4 = 4 b x^3 \dot{x}$. And for the same reason the fluxion of $b x^n = n b x^{n-1} \dot{x}$.

2. By Ex. 5. incr. $\frac{a}{x \dots x} = -\frac{5 a x}{x \dots x}$; suppose x infinitely small, and the increments become the fluxions, and $x, x, x, \&c.$ are each $= x$; therefore flux. $\frac{a}{x^5} = -\frac{5 a \dot{x}}{x^6}$. And for the same reason, flux. $\frac{a}{x^n} = -\frac{n a \dot{x}}{x^{n+1}}$.

3. By

3. By Ex. 14. incr. $xy = x\dot{y} + y\dot{x}$; but x, y being infinitely small, $y, \dot{y}$ are equal; and $x, \dot{y}$ change into the fluxions $\dot{x}, \dot{y}$; therefore flux. $xy = x\dot{y} + y\dot{x}$.

4. By Ex. 12. incr. $\frac{x}{z} = \frac{zx - xz}{z^2}$, and $z, \dot{z}$ being equal, we have flux. $\frac{x}{z} = \frac{zx - xz}{z^2}$.

5. By the same way of reasoning from Examp. 17. The flux. $x^{m+1}z^{n+1} = n+1 \cdot \dot{z} x^{m+1}z^n + m+1 \cdot \dot{x} x^m z^{n+1}$

6. And lastly, proceeding the same way with Examp. 13. The flux. $\frac{1}{x^{m+1}z^{n+1}} = - \frac{n+1 \cdot \dot{z} x^{m+1}z^n + m+1 \cdot \dot{x} x^m z^{n+1}}{x^{m+2}z^{n+2}}$.

P R O P. XII.

To transform a given quantity, or reduce it into another form.

WHEN the quantities concerned in any question are not simple enough, or not in a proper form for operation, they must be reduced to others that are so, by the following

R U L E.

1. In any increment where the successive values are compound quantities; put some single letter for the least compound value, by help of which expunge all the former quantities, and there will come out a simpler expression.

2. To expunge any redundant factor, put in its stead any other factor which is equivalent to it, according to Cor. 1. Prop. I. and this new factor must be either the next preceeding or next succeeding value.

3. When quantities are irregular or deficient, they may be brought to this form $A + Bz + Cz + Dz + \dots$, &c. when z is small, or

to this form $A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \dots$, &c. when z is great, after this manner: instead of any factor in the numerator of the quantity proposed,

proposed, put its equal expressed by some other factor and the increment, according to Prop. I. do the same successively for all other factors you would expunge; and the given quantity will be by degrees brought to the form desired. When the variable quantity is in the denominator, use some of the following ways.

4. When any factor is wanting in the denominator; multiply both numerator and denominator by that factor, then expunge any of these in the numerator as was shewn before.

5. When any defective quantity is given, or when any power of the variable quantity is to be transformed into others containing successive values, or the contrary; assume a series of quantities affected with unknown coefficients, making the number of factors in the greatest term, equal to that in the given one, or equal to the index of the power; then substitute by Prop. III. and multiply the terms at length; then by comparing the homologous terms, the coefficients will be determined. If several quantities or powers be concerned, the same rule must be followed.

EXAMPLE I.

Let $\frac{n}{1} \frac{n}{2}$ be given, where $n=1$. Put $v=n$; then is $v=n+n$, and $v=n=1$. Therefore $\frac{n}{2}=\frac{v}{1}$, and the quantity $\frac{n}{1} \frac{n}{2}$ becomes $\frac{v}{1} v$.

EXAMPLE II.

Given $\frac{3n-1}{1} \times \frac{3n+2}{1}$, where $n=1$, put $v=3n-1$, then $v=3n=3$, and $3n+2=v+3=v+v=v$; whence $\frac{3n-1}{1} \cdot \frac{3n+2}{1}$ is transformed to $\frac{v}{1} v$.

EXAMPLE III.

Let $\frac{1}{3x-1 \cdot 3x+2}$ be given, where $x=1$. Put $3z=3x-1$, then $3x=3z+1$, or $x=z+\frac{1}{3}$. Also $3x+2=3z+3=3xz+1=3xz+\frac{1}{3}=3z$;

therefore $\frac{1}{3x-1 \cdot 3x+2} = \frac{1}{3z \times 3z} = \frac{1}{9z^2}$.

EXAMPLE

EXAMPLE IV.

Let $\frac{1}{2x-1 \cdot 2x}$ be given, where $x=1$. Put $2v = 3x-1$, then $2v=2x$,
or $v=x=1$, and $2x=2v+1$. Therefore $\frac{1}{2x-1 \cdot 2x} = \frac{1}{2v \cdot 2v+1} =$
 $\frac{1}{4x \times \frac{1}{2}} = \frac{1}{4vv} :$

EXAMPLE V.

To reduce $\frac{v}{v \cdot v}$, where v is given. For v put $v-v$; then $\frac{v}{v \cdot v}$
 $= \frac{v-v}{v \cdot v} = \frac{1}{v} - \frac{v}{v \cdot v} .$

Or thus,

Put $v-2v$ for v , then $\frac{v}{v \cdot v} = \frac{v-2v}{v \cdot v} = \frac{1}{v} - \frac{2v}{v \cdot v} .$

EXAMPLE VI.

Let $\frac{x \cdot x}{x \cdot x \cdot x}$ be proposed, put it $= \frac{x}{x \cdot x \cdot x} \times \overline{x+x} = \frac{x}{x \cdot x} + \frac{xx}{xx \cdot x}$

Again, put $\frac{x}{x \cdot x} = \frac{x-2x}{xx} = \frac{1}{x} - \frac{2x}{xx}$. Also, put $\frac{xx}{xx \cdot x} = \frac{x-3x}{xxx} =$
 $= \frac{x}{xx} - \frac{3x^2}{xxx}$. Whence $\frac{x \cdot x}{x \cdot x \cdot x} = \frac{1}{x} - \frac{x}{xx} - \frac{3x^2}{xxx}$.

Otherwise,

$\frac{x \cdot x}{x \cdot x \cdot x} = \frac{x-x \times x+x}{xx \cdot x} = \frac{xx + xx - xx - x^2}{xx \cdot x} = \frac{1}{x} + \frac{x}{xx} - \frac{x}{xx} - \frac{xx}{xxx}$

H

EXAMPLE

EXAMPLE VII.

Let $z z$ be proposed, to bring in z .

1 term $\times \overline{z + 2z} (=z)$	1	$z = z + 2z$ multiply as follows,
2 term $\times z + 3z (=z)$	2	$z z + 2 z z (=z \times z)$
2 + 3	3	$+ 2 z z + 6 z^2 (=2 z \times z)$
	4	$z z = z z + 4 z z + 6 z^2$

EXAMPLE VIII.

To affect the quantity $v v v$ with v .

1 term $\times \overline{v - v} (=v)$	1	$v = v - v$
2 term $\times v - 2v (=v)$	2	$v v - v v$
2 + 3	3	$- v v + 2 v^2$
	4	$v v = v v - 2 v v + 2 v^2$, multiply as below.
1 term $\times \overline{v - v} (=v)$	5	$v v v - v v v$
2 term $\times \overline{v - 2v} (=v)$	6	$- 2 v v v + 4 v^2 v$
3 term $\times \overline{v - 3v} (=v)$	7	$+ 2 v^2 v - 6 v^3$
	8	$v v v = v v v - 3 v v v + 6 v^2 v - 6 v^3$

EXAMPLE IX.

To transform $v v v \dots v$ by the foregoing method.

When $n=1$,

$$v = v + v$$

When $n=2$,

$$v^2 = v v + 2 v v + 2 v^2$$

If

If $n = 3$,

$$\underset{1}{v} \underset{2}{v} \underset{3}{v} = \underset{1}{v} \underset{2}{v} \underset{3}{v} + 3 \underset{1}{v} \underset{2}{v} \underset{3}{v} + 6 \underset{1}{v} \underset{2}{v} \underset{3}{v} + 6 \underset{1}{v} \underset{2}{v} \underset{3}{v}.$$

If $n = 4$,

$$\underset{1}{v} \dots \underset{4}{4} = \underset{1}{v} \dots \underset{3}{v} + 4 \underset{1}{v} \dots \underset{2}{v} \underset{3}{v} + 12 \underset{1}{v} \underset{2}{v} \underset{3}{v} \underset{4}{v} + 24 \underset{1}{v} \underset{2}{v} \underset{3}{v} \underset{4}{v} + 24 \underset{1}{v} \underset{2}{v} \underset{3}{v} \underset{4}{v}. \&c.$$

Whence in general,

$$\underset{1}{v} \underset{2}{v} \underset{3}{v} \dots \underset{n}{v} = \underset{1}{v} \underset{2}{v} \dots \underset{n-1}{v} + n \underset{1}{v} \underset{2}{v} \dots \underset{n-2}{v} \underset{n-1}{v} + n \cdot \overline{n-1} \cdot \underset{1}{v} \underset{2}{v} \dots \underset{n-3}{v} \underset{n-2}{v} \underset{n-1}{v} + n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \underset{1}{v} \underset{2}{v} \dots \underset{n-4}{v} \underset{n-3}{v} \underset{n-2}{v} \underset{n-1}{v} + \&c.$$

Otherwise thus,

When $n=1$,

$$\underset{1}{v} = \underset{2}{v} - \underset{3}{v}$$

If $n = 2$,

$$\underset{1}{v} \underset{2}{v} = \underset{2}{v} \underset{3}{v} - 2 \underset{2}{v} \underset{3}{v} + 2 \underset{3}{v} \underset{4}{v}.$$

If $n = 3$,

$$\underset{1}{v} \underset{2}{v} \underset{3}{v} = \underset{2}{v} \underset{3}{v} \underset{4}{v} - 3 \underset{2}{v} \underset{3}{v} \underset{4}{v} + 6 \underset{3}{v} \underset{4}{v} \underset{5}{v} - 6 \underset{4}{v} \underset{5}{v} \underset{6}{v}.$$

If $n = 4$,

$$\underset{1}{v} \dots \underset{4}{4} = \underset{2}{v} \dots \underset{5}{v} - 4 \underset{2}{v} \dots \underset{3}{v} \underset{5}{v} + 12 \underset{2}{v} \underset{3}{v} \underset{4}{v} \underset{5}{v} - 24 \underset{2}{v} \underset{3}{v} \underset{4}{v} \underset{5}{v} + 24 \underset{3}{v} \underset{4}{v} \underset{5}{v} \underset{6}{v}.$$

And in general,

$$\underset{1}{v} \underset{2}{v} \dots \underset{n}{v} = \underset{2}{v} \underset{3}{v} \dots \underset{n+1}{v} - n \underset{2}{v} \underset{3}{v} \dots \underset{n-1}{v} \underset{n+1}{v} + n \cdot \overline{n-1} \cdot \underset{2}{v} \dots \underset{n-2}{v} \underset{n+1}{v} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \underset{2}{v} \dots \underset{n-3}{v} \underset{n+1}{v} \underset{n+2}{v} + \&c.$$

EXAMPLE X.

To transform the defective quantity $\underset{-3}{z} \underset{1}{z} \underset{4}{z}$, into a perfect one.

$$\begin{aligned} \text{Here } \underset{-3}{z} \underset{1}{z} \underset{4}{z} &= \underset{1}{z} \underset{4}{z} \times \overline{z} - 5 \overline{z} = \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} - 5 \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} = \\ \underset{1}{z} \underset{2}{z} \underset{3}{z} \times \overline{z} + \overline{z} - 5 \underset{1}{z} \underset{2}{z} \underset{3}{z} \times \overline{z} + 2 \overline{z} &= \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} + \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} - 5 \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} \\ - 10 \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z}, \text{ that is, } \underset{-3}{z} \underset{1}{z} \underset{4}{z} &= \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} - 4 \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z} - 10 \underset{1}{z} \underset{2}{z} \underset{3}{z} \underset{4}{z}. \end{aligned}$$

Otherwise

Otherwise thus,

Assume $z z z z = A z z z z + B z z z + C z z + D z$, then
for z, z, z, z, z , put $z - 3z, z + z, z + 2z, z + 3z$, and
 $z + 4z$, respectively; and the quantities being multiplied, you have

$$z^4 + 2z^3z - 11z^2z^2 - 12zzz^3 = Az^4 + 6Az^3z + 11Az^2z^2 + 6Az^3z^3 \\ + Bz^3 + 3Bz^2z + 2Bzz^2 \\ + Cz^2 + Cz \\ + Dz$$

Then comparing the homologous terms, $A=1, B=2z-6Az=-4z,$
 $C=-11z^2-11Az^2-3Bz=-10z^2, D=-12z^3-6Az^3-2Bz^2-Cz=0,$
whence $z z z z = z z z z - 4 z z z z - 10 z z z^2.$

Or rather thus,

Put $z z z z = A z z z z + B z z z + C z z + D z$, then
putting $z-z, z-2z, \&c.$ for $z, z, \&c.$ and multiplying, we have

$$z^4 + 2z^3z - 11z^2z^2 - 12zzz^3 = Az^4 - 6Az^3z + 11Az^2z^2 - 6Az^3z^3 \\ + Bz^3 - 3Bz^2z + 2Bzz^2 \\ + C - Cz \\ + D$$

And equating the coefficients $A=1, B=2z+6Az=8z;$
 $C=-11z^2-11Az^2+3Bz=2z^2, D=-12z^3+6Az^3-2Bz^2+Cz=-20z^3.$ Whence
 $z z z z = z z z z + 8 z z z z + 2 z^2 z z - 20 z^3 z.$

EXAMPLE XI.

Let $\frac{a}{vv}$ be given. Then $\frac{a}{vv} = \frac{av}{vvv} = \frac{av-av}{vvv} = \frac{a}{vv} - \frac{av}{vvv}.$

EXAMPLE

E X A M P L E XII.

To transform $\frac{1}{z}$, $\frac{1}{z z}$, $\frac{1}{z z z}$, &c. Here $\frac{1}{z} = \frac{z}{z z} =$
 $\frac{z-z}{z z} = \frac{1}{z} - \frac{z}{z z}$. Also $\frac{1}{z z} = \frac{z}{z z z} = \frac{z-2z}{z z z} = \frac{1}{z z} -$
 $\frac{z z}{z z z}$.

After the same manner,

$$\frac{1}{z z z} = \frac{1}{z z z} - \frac{3z}{z z z z}$$

$$\frac{1}{z \dots z} = \frac{1}{z \dots z} - \frac{4z}{z \dots z}$$

And in general,

$$\frac{1}{z \dots z} = \frac{1}{z \dots z} - \frac{n z}{z \dots z}$$

E X A M P L E XIII.

Let $\frac{1}{v v}$ be given. Suppose $\frac{1}{v v} = \frac{A}{v v} + \frac{B}{v v v} + \frac{C}{v \dots v}$,

reducing them to a common denominator, we have

$$\frac{v v}{v \dots v} = \frac{A \times v v + B v + C}{v \dots v}, \text{ that is, } v \times v + v = A \times v + 2v \times v + 3v +$$

$$B \times v + 3v + C. \text{ When these are multiplied we have}$$

$$v v + v v = A v v + 5A v v + 6A v^2$$

$$+ B v + 3B v$$

$$+ C$$

I

Then

Then $A = 1$. $B = v - 5Av = -4v$. $C = -6Av^2 - 3Bv = 6v^2$.

And therefore $\frac{1}{v \underset{2}{v} \underset{3}{v}} = \frac{1}{v \underset{1}{v}} - \frac{4v}{v \underset{1}{v} \underset{2}{v}} + \frac{6v^2}{v \dots \underset{3}{v}}$.

Likewise $\frac{1}{v \underset{2}{v} \underset{3}{v} \underset{4}{v}} = \frac{1}{v \underset{1}{v} \underset{2}{v}} - \frac{6v}{v \dots \underset{3}{v}} + \frac{12v^2}{v \dots \underset{4}{v}}$, &c.

EXAMPLE XIV.

Let $\frac{1}{v \underset{3}{v} \underset{5}{v}}$ be proposed. This becomes $\frac{v \underset{1}{v} \underset{2}{v}}{v \dots \underset{5}{v}}$. Reduce the numerator to such quantities whose highest factor may be v ; thus, $v \underset{1}{v} \underset{2}{v} = v \underset{2}{v} \times v - 4v = v \underset{2}{v} \underset{4}{v} - 4v \underset{2}{v} \underset{4}{v} = v \underset{4}{v} \times v - v - 4v \underset{4}{v} \times v - 3v = v \underset{3}{v} \underset{4}{v} \underset{5}{v} - v \underset{4}{v} \underset{5}{v} - 4v \underset{4}{v} \underset{5}{v} + 12v^2 \underset{4}{v} = v \underset{3}{v} \underset{4}{v} \underset{5}{v} - 5v \underset{4}{v} \underset{5}{v} + 12v^2 \times v - v = v \underset{3}{v} \underset{4}{v} \underset{5}{v} - 5v \underset{4}{v} \underset{5}{v} + 12v^2 \underset{5}{v} - 12v^3$; therefore $\frac{1}{v \underset{3}{v} \underset{5}{v}} = \frac{v \underset{3}{v} \underset{4}{v} \underset{5}{v} - 5v \underset{4}{v} \underset{5}{v} + 12v^2 \underset{5}{v} - 12v^3}{v \dots \underset{5}{v}} = \frac{1}{v \underset{1}{v} \underset{2}{v}} - \frac{5v}{v \dots \underset{3}{v}} + \frac{12v^2}{v \dots \underset{4}{v}} - \frac{12v^3}{v \dots \underset{5}{v}}$.

EXAMPLE XV.

Let $\frac{1}{v \underset{1}{v}}$, $\frac{1}{v \underset{2}{v}}$, $\frac{1}{v \underset{3}{v}}$, &c. be given. Here $\frac{1}{v \underset{1}{v}} = \frac{v}{v \underset{1}{v}} = \frac{v-v}{v \underset{1}{v}} = \frac{1}{v} - \frac{v}{v \underset{1}{v}}$.

Otherwise,

By Prop. III. $m=1$; therefore $\frac{1}{v \underset{1}{v}} = \frac{1}{v} - \frac{m}{v} Av = \frac{1}{v} -$

$\frac{v}{v \underset{1}{v}}$. And by the same Proposition we have likewise

$$\frac{1}{v^2} = \frac{1}{v} - \frac{2v}{v^2 v} + \frac{2v^2}{v^2 v^2 v}.$$

$$\text{And } \frac{1}{v^3} = \frac{1}{v} - \frac{3v}{v^2 v} + \frac{6v^2}{v^2 v^2 v} - \frac{6v^3}{v^2 \dots v}, \&c.$$

E X A M P L E X V I.

Let $\frac{1}{x^4}$ be proposed. By Prop. III. Cor. 2. $\frac{1}{x^4} = \frac{1}{x^3} -$
 $\frac{m-1}{x^2} A x - \frac{m-2}{x} B x, \&c. = \frac{1}{x^3} - \frac{3}{x^2} A x -$
 $\frac{2}{x} B x - \frac{1}{x} C x = \frac{1}{x^3} - \frac{3x}{x^2 x} + \frac{32x^2}{x \dots x} - \frac{6x^3}{x \dots x}.$

E X A M P L E X V I I.

Let $\frac{ax}{x^4}$ be given. Then by Cor. 1. Prop. I. $\frac{ax}{x^4} = \frac{x-x}{x^4} a =$
 $\frac{a}{x^3} - \frac{ax}{x^4}.$

But by Cor. 2. Prop. III. putting $m=4,$

$$\frac{ax}{x^4} = \frac{ax}{x^3} - \frac{2x}{x} A - \frac{x}{x} B = \frac{ax}{x^3} - \frac{2ax^2}{x \dots x} + \frac{2ax^3}{x \dots x}.$$

Whence $\frac{ax}{x^4} = \frac{a}{x^3} - \frac{ax}{x^3} + \frac{2ax^2}{x \dots x} - \frac{2ax^3}{x \dots x},$ the quantity required.

Or

Or thus,

$$\begin{aligned} \frac{ax}{x \cdot x \cdot x} &= \frac{ax - ax}{x \cdot x \cdot x} = \frac{a}{x \cdot x} - \frac{ax}{x \cdot x \cdot x} = \frac{a}{x \cdot x} - \frac{ax \cdot x \cdot x}{x \cdot x \cdot x \cdot x} = \frac{a}{x \cdot x} \\ \frac{ax \times x + 2x \times x - x}{x \cdot x \cdot x} &= \frac{a}{x \cdot x} - \frac{ax \cdot x \cdot x - ax^2x + 2ax^2x - 2ax^3}{x \cdot x \cdot x \cdot x} \\ &= \frac{a}{x \cdot x} - \frac{ax}{x \cdot x \cdot x} + \frac{ax^2}{x \cdot x \cdot x \cdot x} - \frac{2ax^2}{x \cdot x \cdot x \cdot x} + \frac{2ax^3}{x \cdot x \cdot x \cdot x} \end{aligned}$$

EXAMPLE XVIII.

To transform $\frac{z}{z \cdot z}$, $\frac{z}{z \cdot z \cdot z}$, $\frac{z}{z \cdot \dots \cdot z}$, &c.

$$1. \frac{z}{z \cdot z} = \frac{z - z}{z \cdot z} = \frac{1}{z} - \frac{z}{z \cdot z}$$

$$2. \frac{z}{z \cdot z \cdot z} = \frac{z - 2z}{z \cdot z \cdot z} = \frac{1}{z \cdot z} - \frac{2z}{z \cdot z \cdot z}$$

$$3. \frac{z}{z \cdot \dots \cdot z} = \frac{z - 3z}{z \cdot \dots \cdot z} = \frac{1}{z \cdot \dots \cdot z} - \frac{3z}{z \cdot \dots \cdot z}$$

And in general,

$$\frac{z}{z \cdot \dots \cdot z} = \frac{z - nz}{z \cdot \dots \cdot z} = \frac{1}{z \cdot \dots \cdot z} - \frac{nz}{z \cdot \dots \cdot z}$$

EXAMPLE XIX.

To transform $\frac{z \cdot z}{z \cdot z}$, $\frac{z \cdot z}{z \cdot z \cdot z}$, $\frac{z \cdot z}{z \cdot \dots \cdot z}$, &c.

$$1. \frac{z \cdot z}{z \cdot z} = \frac{z - z \times z}{z \cdot z} = 1 - \frac{z \cdot z}{z \cdot z} = 1 - \frac{z}{z} + \frac{z \cdot z}{z \cdot z}$$

$$\begin{aligned}
 2. \frac{z z}{z z z} &= \frac{\overline{z-2z} \times \overline{z-z}}{z z z} = \frac{z z - 2 z z - z z + 2 z^2}{z z z} = \\
 &= \frac{1}{z} - \frac{3 z}{z z} + \frac{4 z z}{z z z}. \\
 3. \frac{z z}{z \dots z} &= \frac{\overline{z-3z} \times \overline{z-2z}}{z \dots z} = \frac{z z - 2 z z - 3 z \times \overline{z-z} + 6 z z}{z \dots z} \\
 &= \frac{1}{z z} - \frac{5 z}{z \dots z} + \frac{9 z z}{z \dots z}, \&c.
 \end{aligned}$$

In general,

$$\begin{aligned}
 \frac{z z}{z \dots z} &= \frac{\overline{z-nz} \times \overline{z-n-1 \cdot z}}{z \dots z} = \frac{z z - n-1 \cdot z z - n z \times \overline{z-z}}{z \dots z} \\
 &+ \frac{n \cdot n-1 \cdot z^2}{z \dots z} = \frac{1}{z \dots z} - \frac{2n-1 \cdot z}{z \dots z} + \frac{n n z z}{z \dots z}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{z z}{z z} &= 1 - \frac{z}{z} + \frac{z z}{z z}, \\
 \frac{z z}{z z z} &= \frac{1}{z} - \frac{3 z}{z z} + \frac{4 z z}{z z z}, \\
 \frac{z z}{z \dots z} &= \frac{1}{z z} - \frac{5 z}{z \dots z} + \frac{9 z z}{z \dots z}, \&c.
 \end{aligned}$$

In general,

$$\frac{z z}{z \dots z} = \frac{1}{z \dots z} - \frac{2n-1 \cdot z}{z \dots z} + \frac{n n z z}{z \dots z}.$$

K

EXAMPLE

EXAMPLE XX.

To transform any power v , vv , v^3 , &c. into quantities containing the successive values of v .

	1	$v = v$
$1 \times v + v (=v)$	2	$vv = vv + vv$
1 term $\times v + 2v (=v)$	3	$vvv + 2vvv$
2 term $\times v + v (=v)$	4	$+ vvv + vv^2$
3 + 4	5	$v^3 = vv + 3vv + vv^2$
1 term $\times v + 3v (=v)$	6	$vvv + 3vvv$
2 term $\times v + 2v (=v)$	7	$+ 3vvv + 6vvv^2$
3 term $\times v + v (=v)$	8	$+ vv^2 + vv^3$
6 + 7 + 8	9	$v^4 = v \dots v + 6v \dots vv + 7vvv^2 + vv^3$
		&c.

After the same manner,

$$vv = vv - vv$$

$$v^3 = vv - 3vv + vv^2$$

$$v^4 = v \dots v - 6vvv + 7vvv^2 - vv^3, \text{ \&c.}$$

More generally thus,

Let the following Table I. be constructed after this manner: Multiply any number in any horizontal row, by the number of that row, and add the number above it, gives the next number on the right hand in the same horizontal row: Thus

$$65 \times 4 + 90 = 350.$$

TABLE

T A B L E I.

1	1	1	1	1	1	1	1	1
	1	3	7	15	31	63	127	255
		1	6	25	90	301	966	3025
			1	10	65	350	1701	7770
				1	15	140	1050	6951
					1	21	266	2640
						1	28	462
							1	36
								1

Then to reduce the powers to the form of successive values, take the numbers in the ascending columns for the coefficients, and you have the values of the several powers :

Thus,

$$v = v$$

$$v^2 = v \underset{-1}{v} + v \underset{-1}{v}$$

$$v^3 = v \underset{-1}{v} \underset{-2}{v} + 3 v \underset{-1}{v} \underset{-1}{v} + v \underset{-1}{v}^2$$

$$v^4 = v \underset{-1}{v} \underset{-2}{v} \underset{-3}{v} + 6 v \underset{-1}{v} \underset{-2}{v} \underset{-1}{v} + 7 v \underset{-1}{v} \underset{-1}{v}^2 + v \underset{-1}{v}^3$$

$$v^5 = v \dots \underset{-4}{v} + 10 v \dots \underset{-3}{v} \underset{-2}{v} + 25 v \dots \underset{-2}{v} \underset{-1}{v}^2 + 15 v \underset{-1}{v} \underset{-1}{v}^3 + v \underset{-1}{v}^4$$

&c.

E X A M P L E XXI.

To transform any fractional power $\frac{1}{v}$, $\frac{1}{v^2}$, $\frac{1}{v^3}$, $\frac{1}{v^4}$, &c. into quantities, whose denominators contain the successive values of v .

Suppose $\frac{1}{v^2} = \frac{A}{v \underset{1}{v}} + \frac{B}{v \underset{1}{v} \underset{2}{v}} + \frac{C}{v \dots \underset{3}{v}}$, &c. that is,

$v \dots \underset{3}{v} = v^2 \times : A \underset{2}{v} \underset{3}{v} + B \underset{3}{v} + C$, then putting $v+v$, $v+2v$, and $v+3v$, for $v \underset{1}{v}$, $v \underset{2}{v}$, $v \underset{3}{v}$, and then multiplying, we have

$$v^4 + 6v^3 \cdot v + 11v^2 \cdot v^2 + 6v \cdot v^3 = Av^4 + 5Av^3 \cdot v + 6Av^2 \cdot v^2 + Bv^3 + 3Bv^2 \cdot v + C$$

Then equating the homologous terms, $A=1$. $B=6v - 5Av = v$. $C=$

$$11v^2 - 6Av^2 - 3Bv = 2v^2, \text{ \&c. } \text{ And } \frac{1}{v \cdot v} = \frac{1}{v \cdot v} + \frac{v}{v \cdot v \cdot v} + \frac{2v^2}{v \cdot \dots \cdot v}, \text{ \&c. }$$

$$\text{And likewise } \frac{1}{v^3} = \frac{1}{v \cdot v \cdot v} + \frac{3v}{v \cdot \dots \cdot v} + \frac{11v^2}{v \cdot \dots \cdot v} + \frac{50v^3}{v \cdot \dots \cdot v}, \text{ \&c. }$$

$$\text{And } \frac{1}{v^4} = \frac{1}{v \cdot \dots \cdot v} + \frac{6v}{v \cdot \dots \cdot v} + \frac{35v^2}{v \cdot \dots \cdot v}, \text{ \&c. }$$

And after the same manner,

$$\frac{1}{v \cdot v} = \frac{1}{v \cdot v} - \frac{1v}{v \cdot \dots \cdot v} + \frac{2v^2}{v \cdot \dots \cdot v} - \frac{6v^3}{v \cdot \dots \cdot v} \text{ \&c. }$$

$$\frac{1}{v^3} = \frac{1}{v \cdot v \cdot v} - \frac{3v}{v \cdot \dots \cdot v} + \frac{11v^2}{v \cdot \dots \cdot v} - \text{ \&c. }$$

$$\frac{1}{v^4} = \frac{1}{v \cdot \dots \cdot v} - \frac{6v}{v \cdot \dots \cdot v} + \frac{35v^2}{v \cdot \dots \cdot v} - \text{ \&c. }$$

More generally thus,

Let the following Table II. be thus constructed: Multiply any number in any horizontal row, by the number of that row, and add the next number on the left hand, gives the number below. Thus,

$$35 \times 5 + 50 = 225.$$

TABLE

TABLE II.

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6760	1960	322	28	1		
40320	109584	118124	67284	22449	4536	546	36	1	

Then to reduce the powers to the forms required, take the numbers in the descending columns for the coefficients of the several terms :
Thus,

$$\frac{1}{v^2} = \frac{1}{v^2} + \frac{1v}{v^2 \cdot v} + \frac{2v^2}{v \dots v} + \frac{6v^3}{v \dots v} \&c.$$

$$\frac{1}{v^3} = \frac{1}{v^3} + \frac{3v}{v^3 \cdot v} + \frac{11v^2}{v \dots v} + \frac{50v^3}{v \dots v} \&c.$$

$$\frac{1}{v^4} = \frac{1}{v^4} + \frac{6v}{v^4 \cdot v} + \frac{35v^2}{v \dots v} + \frac{225v^3}{v \dots v} \&c.$$

$$\frac{1}{v^5} = \frac{1}{v^5} + \frac{10v}{v^5 \cdot v} + \frac{85v^2}{v \dots v} + \&c.$$

EXAMPLE XXII.

To transform $\alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x$ into quantities of successive values: and the contrary.

Suppose it = $A x \dots x + B x \dots x + C x \dots x + D x \dots x + E x$;

Then by Table I. Ex. 18, we have

L

$$\begin{aligned}
 \alpha x^5 &= \alpha x \dots x_{-4} + 10 \alpha x \dots x_{-3} x + 25 \alpha x \dots x_{-2} x^2 + 15 \alpha x \dots x_{-1} x^3 + \alpha x x^4 \\
 \beta x^4 &= \beta x \dots x_{-3} + 6 \beta x \dots x_{-2} x + 7 \beta x x x^2 + \beta x x^3 \\
 \gamma x^3 &= \gamma x \dots x_{-2} + 3 \gamma x x x + \gamma x x^2 \\
 \delta x^2 &= \delta x x + \delta x x \\
 \epsilon x &= \epsilon x \\
 &= A x \dots x_{-4} + B x \dots x_{-3} + C x \dots x_{-2} + D x x + E x
 \end{aligned}$$

Then comparing the terms, $A = \alpha$, $B = \beta + 10\alpha x$, $C = \gamma + 6\beta x + 25\alpha x^2$, &c. Whence

$$\begin{aligned}
 \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x &= \alpha x \dots x_{-4} + \beta x \dots x_{-3} + \gamma x \dots x_{-2} + \delta x x + \epsilon x \\
 &= \left. \begin{array}{l} \alpha x \dots x_{-4} \\ \beta x \dots x_{-3} \\ \gamma x \dots x_{-2} \\ \delta x x \\ \epsilon x \end{array} \right\} x \dots x_{-4} + \left. \begin{array}{l} 10\alpha x \\ 6\beta x \\ 25\alpha x^2 \end{array} \right\} x \dots x_{-3} + \left. \begin{array}{l} \gamma \\ 3\beta x \\ 7\alpha x^2 \end{array} \right\} x \dots x_{-2} + \left. \begin{array}{l} \delta \\ 3\gamma x \\ 7\beta x^2 \\ 15\alpha x^3 \end{array} \right\} x x_{-1} + \left. \begin{array}{l} \epsilon \\ \delta x \\ \gamma x^2 \\ \beta x^3 \\ \alpha x^4 \end{array} \right\} x.
 \end{aligned}$$

Where the numbers are those in the horizontal columns, Tab. I.

And if $\alpha x^5 + \beta x^4 + \gamma x^3$, &c. was to be transformed into $A x \dots x_{-4} + B x \dots x_{-3} + C x \dots x_{-2}$, &c. the series would be the same as this, only the signs of the quantities in each coefficient are alternately affirmative and negative.

On the contrary,

$A x \dots x_{-4} + B x \dots x_{-3} + C x \dots x_{-2} + D x x + E x$, being given; we shall have $\alpha = A$, $\beta = B - 10\alpha x = B - 10A x$, $\gamma = C - 25\alpha x^2 - 6\beta x = C - 6B x + 35A x^2$, &c. whence at last

$$A x \dots x_{-4} + B x \dots x_{-3} + C x \dots x_{-2} + D x x + E x = A x^5 + \left. \begin{array}{l} B \\ -10A x \end{array} \right\} x^4 + C$$

+ C

$$\left. \begin{array}{l} + C \\ - 6 B x \\ + 35 A x^2 \end{array} \right\} x^3 \quad \left. \begin{array}{l} + D \\ - 3 C x \\ + 11 B x^2 \\ - 50 A x^3 \end{array} \right\} x^2 \quad \left. \begin{array}{l} + E \\ - D x \\ + 2 C x^2 \\ - 6 B x^3 \\ + 24 A x^4 \end{array} \right\} x, \text{ where}$$

the numeral coefficients are found in the descending columns of Table II.

And if $A x \dots x + B x \dots x + C x \dots x$, &c. was to be transformed into the powers of x ; the series would be the same as this, only all the signs affirmative.

EXAMPLE XXIII.

To transform $\frac{\alpha}{x x} + \frac{\beta}{x^3} + \frac{\gamma}{x^4} + \frac{\delta}{x^5} + \frac{\epsilon}{x^6}$, &c. into quantities containing successive values: and on the contrary.

$$\text{Suppose it} = \frac{A}{x x} + \frac{B}{x x x} + \frac{C}{x \dots x} + \frac{D}{x \dots x} + \frac{E}{x \dots x} \text{ \&c.}$$

Then by Table II. Exam. 19, we have

$$\begin{aligned} \frac{\alpha}{x x} &= \frac{\alpha}{x x} + \frac{\alpha x}{x x x} + \frac{2 \alpha x^2}{x \dots x} + \frac{6 \alpha x^3}{x \dots x} + \frac{24 \alpha x^4}{x \dots x} \\ \frac{\beta}{x^3} &= \frac{\beta}{x x x} + \frac{3 \beta x}{x \dots x} + \frac{11 \beta x^2}{x \dots x} + \frac{50 \beta x^3}{x \dots x} \\ \frac{\gamma}{x^4} &= \frac{\gamma}{x \dots x} + \frac{6 \gamma x}{x \dots x} + \frac{35 \gamma x^2}{x \dots x} \\ \frac{\delta}{x^5} &= \frac{\delta}{x \dots x} + \frac{10 \delta x}{x \dots x} \\ \frac{\epsilon}{x^6} &= \frac{\epsilon}{x \dots x} \end{aligned}$$

$$= \frac{A}{x x} + \frac{B}{x x x} + \frac{C}{x \dots x} + \frac{D}{x \dots x} + \frac{E}{x \dots x}$$

And

And equating the like terms, $A=\alpha$, $B=\beta + \alpha x$, $C=\gamma + 3\beta x + 2\alpha x^2$, &c. Whence

$$\frac{\alpha}{x x} + \frac{\beta}{x^3} + \frac{\gamma}{x^4} + \frac{\delta}{x^5} + \frac{\varepsilon}{x^6}, \&c. = \frac{\alpha}{x x} + \frac{\beta}{x} \left\{ \frac{1}{x x x} \right. \\ \left. + \gamma \right\} \frac{1}{x \dots x} + \frac{\delta}{x} \left\{ \frac{1}{x \dots x} \right. + \frac{\varepsilon}{x} \left\{ \frac{1}{x \dots x} \right. \\ \left. + 3\beta x \right\} \frac{1}{x \dots x} + 6\gamma x \left\{ \frac{1}{x \dots x} \right. + 10\delta x \left\{ \frac{1}{x \dots x} \right. \\ \left. + 2\alpha x^2 \right\} \frac{1}{x \dots x} + 11\beta x^2 \left\{ \frac{1}{x \dots x} \right. + 35\gamma x^2 \left\{ \frac{1}{x \dots x} \right. \\ \left. + 6\alpha x^3 \right\} \frac{1}{x \dots x} + 50\delta x^3 \left\{ \frac{1}{x \dots x} \right. \\ \left. + 24\alpha x^4 \right\} \frac{1}{x \dots x}, \&c.$$

Where the numeral coefficients are the same as the horizontal rows taken backward, in Tab. II.

If $\frac{\alpha}{x x} + \frac{\beta}{x^3} + \frac{\gamma}{x^4}, \&c.$ was transformed into $\frac{A}{x x} + \frac{B}{x \dots x} + \frac{C}{x \dots x}, \&c.$ the series would be the same, only the signs of the quantities in the several coefficients, would be $+$ and $-$ alternately.

On the contrary,

If $\frac{A}{x x} + \frac{B}{x x x} + \frac{C}{x \dots x} + \frac{D}{x \dots x} + \&c.$ was given, to be transformed into the powers of x ; we have $\alpha=A$, $\beta=B-\alpha x=B-Ax$, $\gamma=C-3Bx+Ax^2$, &c. Whence

$$\frac{A}{x x} + \frac{B}{x x x} + \frac{C}{x \dots x} + \frac{D}{x \dots x} + \frac{E}{x \dots x}, \&c. = \frac{A}{x x} + \\ \left\{ \frac{B}{x} - \frac{A}{x} \right\} \frac{1}{x^3} + \left\{ \frac{C}{x} - \frac{3B}{x} + \frac{A}{x} \right\} \frac{1}{x^4} + \left\{ \frac{D}{x} - \frac{6C}{x} + \frac{7B}{x} - \frac{A}{x} \right\} \frac{1}{x^5} + \left\{ \frac{E}{x} - \frac{10D}{x} + \frac{25C}{x} - \frac{15B}{x} + \frac{A}{x} \right\} \frac{1}{x^6}, \&c.$$

Where the numbers in the coefficients are the ascending numbers of Tab. I.

If

If $\frac{A}{x \underset{-1}{x}} + \frac{B}{x \dots x \underset{-2}{x}} + \frac{C}{x \dots x \underset{-3}{x}}; \&c.$ was transformed into the powers of x , the coefficients will all be the same, only all the terms are affirmative.

P R O P. XIII.

To find the integral of any given increment.

R U L E.

1. **I**F the variable quantity increases uniformly, and the integral proposed consists of the successive values of it multiplied together, or is a perfect increment not fractional: *Multiply the given increment by the next preceeding value of the variable quantity, then divide by the new number of factors, and the constant increment.*

2. In a *fractional quantity*, where the variable quantity increases uniformly, and the denominator is perfect, containing the successive values of the variable quantity: *Throw out the greatest value of the variable quantity, then divide by the new number of factors, and by the constant increment with a negative sign.*

3. In *compound quantities*, let one part be denoted by X , the other by V ; then the integral of XV is had by this series, $[XV] = [X] V - [X] V + [X] V + [X] V + \&c.$ by Prop. IX.

4. If the quantity be a *deficient increment*, it must be transformed into one or more perfect ones, by Prop. XII. and then the integral is found by Art. 1, 2, before.

5. When the increment is *any power* of the variable quantity; reduce it into perfect quantities, containing the successive values, by Art. 5. Prop. XII. then the increment is found by Art. 1. or 2. before. Or thus,

6. When the increment is *any power, or sums of powers*, of the variable quantity; for its integral, assume a series of powers, with indetermined coefficients, making the index of the highest power greater by 1 than the index of the highest power of the increment; then finding the increment of each term, by Prop. VIII. put the sum of all equal

to the given increment; then by comparing the homologous terms, the coefficients will be determined.

7. Sometimes the integral may be had *by comparing* the given increment with some other increments that have been obtained before by Prop. XI. and if the given increment agree with any of these, the integral of that is the integral required. For the ready comparing these increments, you have several forms of increments and integrals, in a Table afterwards.

8. Sometimes the integral may be found, by *assuming* other variable quantities to make up the integral; and then finding their values, by help of the *given* equation, and its increment.

9. When any *doubt* arises about the integral, find its increment by Prop. XI; and if it be the same with the given one; the integral is rightly found, otherwise not.

EXAMPLE I.

To find the integral of x ; the integral is either x , $\frac{x}{1}$, $\frac{x}{2}$, $\frac{x}{3}$, &c. or $\frac{x}{-1}$, $\frac{x}{-2}$, $\frac{x}{-3}$, &c. supposing x to increase uniformly; but the integral is only x , when x is variable.

EXAMPLE II.

What is the integral of $\frac{x}{x}$, supposing x to increase uniformly.

By the general rule Art. 2. the integral of $\frac{x}{x}$, $\frac{x}{x}$, $\frac{x}{x}$, $\frac{x}{x}$, $\frac{x}{x}$, &c.

&c. is $\frac{x}{0x} = \text{infinity}$.

And to find the integral of $\frac{x}{x}$, and x not given. Suppose there is a set of geometrical proportionals, b , $\overline{a+1} \cdot b$, $\overline{a+1}^2 \cdot b$, $\overline{a+1}^3 \cdot b$, ... $\overline{a+1}^{n+1} b$, call any one of the terms z ; as $\overline{a+1} b$. Then $z = \overline{a+1}^{n+1} b - \overline{a+1}^n b = \overline{a+1}^n b \times a = az$, whence $\frac{z}{z} = a$. Therefore if $\frac{z}{z} = a$, the integral is $z = \overline{a+1}^n b$.

EXAMPLE

E X A M P L E I I I .

To find the integral of $a z z z + b z z z z + C z z z z$. By

$$\text{Art. I. the int.} = \frac{a z z z z}{4z} + \frac{b z z z z z}{5z} + \frac{c z z z z z}{5z}$$

E X A M P L E I V .

The integral of $\frac{a}{z z z} + \frac{b}{z z z z} + \frac{c}{z z z z}$, is $-\frac{a}{2 z z z}$
 $-\frac{b}{3 z z z z} - \frac{c}{2 z z z z}$.

E X A M P L E V .

To find the integral of $z z z z$, this quantity being defective, will be reduced (by Prop. XII.) to this $z z z z - 4 z z z z - 10 z z z z$,

whose integral is $\frac{z z z z}{5z} - \frac{4 z z z z}{4z} - \frac{10 z z z z}{3z} = \frac{z z z z}{5z}$
 $-\frac{z z z z}{3} - \frac{10}{3} z z z z$.

E X A M P L E V I .

To find the integral of $\frac{a}{v v}$. This transformed by Prop. XII. becomes $\frac{a}{v v} - \frac{a v}{v v v}$, here v is given; then its integral is $-\frac{a}{v v}$

$$+ \frac{a}{2 v v}$$

E X A M P L E

EXAMPLE VII.

The integral of v^3 is required, where v is given; this (by Prop. XII.)

is reduced to $v \frac{v}{1} \frac{v}{2} - 3 v \frac{v}{1} \frac{v}{2} + v \frac{v^2}{2}$, whose integral is $\frac{v \dots v}{4v}$

$$- \frac{3 \frac{v \dots v}{1}}{3} + \frac{\frac{v v v}{1}}{2}$$

Otherwise,

Suppose $[v^3] = Av^4 + Bv^3 + Cv^2 + Dv$, then taking the increments,

$$\begin{aligned} v^3 &= 4A v^3 v + 6A v^2 v^2 + 4A v v^3 + A v^4 \\ &\quad + 3B v^2 v + 3B v v^2 + B v^3 \\ &\quad + 2C v v + C v^2 \\ &\quad + D v \end{aligned}$$

Then equating the coefficients of the homologous terms, $4Av=1$, and

$$A = \frac{1}{4v}; \quad 3Bv = -6Av^2, \text{ and } B = -\frac{1}{2} \cdot 2Cv = -3Bv^2 - 4Av^3, \text{ and } C = \frac{1}{4}v; \quad D = -Av^3 - Bv^2 - Cv, = 0.$$

$$\text{Whence the integral of } v^3 = \frac{v^4}{4v} - \frac{v^3}{2} + \frac{v^2 v}{4}.$$

EXAMPLE VIII.

To find the integral of $\frac{a}{v^3}$. By Prop. XII. $\frac{1}{v^3}$ is transformed

$$\text{into } \frac{1}{v \frac{v}{1} \frac{v}{2}} + \frac{3v}{v \dots v} + \frac{11v^2}{v \dots v} + \frac{50v^3}{v \dots v} + \&c. \text{ whose in-}$$

tegral

tegral is $-\frac{1}{2v \cdot v \cdot v} - \frac{3v}{3v \cdot \dots \cdot v \cdot v} - \frac{11v^2}{4v \cdot \dots \cdot v \cdot v} \&c.$ and the

increment of $\frac{a}{v^3} = -\frac{a}{2v \cdot v \cdot v} - \frac{3a}{3v \cdot \dots \cdot v} - \frac{11av}{4v \cdot \dots \cdot v} - \frac{10av^2}{v \cdot \dots \cdot v} \&c.$

Otherwise,

Suppose $\left[\frac{a}{v^3}\right] = \frac{A}{v \cdot v} + \frac{B}{v^3} + \frac{C}{v^4} + \frac{D}{v^5} \&c.$ then taking the increment

$$\begin{aligned} av^{-3} = & -2Av \cdot v^{-3} + 3A \cdot v^2 v^{-4} - 4A \cdot v^3 v^{-5} + 5A \cdot v^4 v^{-6} \&c. \\ & - 3B \cdot v + 6B \cdot v^2 - 10B \cdot v^3 \&c. \\ & - 4C \cdot v + 10C \cdot v^2 \&c. \\ & - 5D \cdot v \&c. \end{aligned}$$

And equating the homologous terms, $-2Av = a$, and $A = \frac{-a}{2v}$;

also $3B \cdot v = 3A \cdot v^2$, and $B = -\frac{a}{2}$; in like manner $C = \frac{-av}{4}$,

and $D = 0$, &c. whence the int. $\frac{a}{v^3} = \frac{-a}{2v \cdot v^2} - \frac{a}{2v^3} - \frac{av}{4v^4} \&c.$

EXAMPLE IX.

To find the integral of $ax + bx^2 + cx^3 + dx^4 \&c.$

By Prop. XII. this quantity will be transformed as follows :

$$a x = a x$$

$$b x^2 = b x x + b x x_{-1}$$

$$c x^3 = c x^2 x + 3 c x x_{-1} + c x x x_{-2}$$

$$d x^4 = d x^3 x + 7 d x^2 x x_{-1} + 6 d x x x x_{-2} + d x x x x_{-3}$$

&c. that is,

$$+ \begin{array}{l} a x \\ b x^2 \\ \text{\&c.} \end{array} = \begin{array}{l} a \\ + b x \\ + c x^2 \\ + d x^3 \end{array} \left\{ \begin{array}{l} x \\ x x \\ x x x \\ \text{\&c.} \end{array} \right. + \begin{array}{l} b \\ + 3 c x \\ + 7 d x^2 \end{array} \left\{ \begin{array}{l} x x \\ x x x \\ \text{\&c.} \end{array} \right. + \begin{array}{l} c \\ + 6 d x \end{array} \left\{ \begin{array}{l} x x x \\ x x x x \\ \text{\&c.} \end{array} \right. + d x \dots x_{-3}$$

And therefore the int. of $a x + b x^2 + c x^3 + d x^4$ &c. =

$$+ \begin{array}{l} a \\ b x \\ c x^2 \\ d x^3 \\ \text{\&c.} \end{array} \left\{ \begin{array}{l} x x \\ x x x \\ x x x x \\ \text{\&c.} \end{array} \right. \frac{1}{2 x} + \begin{array}{l} b \\ + 3 c x \\ + 7 d x^2 \\ \text{\&c.} \end{array} \left\{ \begin{array}{l} x x x \\ x x x x \\ \text{\&c.} \end{array} \right. \frac{1}{3 x^2} + \begin{array}{l} c \\ + 6 d x \end{array} \left\{ \begin{array}{l} x x x x \\ x x x x x \\ \text{\&c.} \end{array} \right. \frac{1}{4 x^3} + \begin{array}{l} d x \dots x_{-3} \\ \text{\&c.} \end{array} \frac{1}{5 x^4} \quad \text{\&c.}$$

where the numbers in the several coefficients are the same as the horizontal rows in Tab. I.

EXAMPLE X.

To find the integral of $a x^n + b x^{n-1} + c x^{n-2} + d x^{n-3}$ &c.

Let it be $A x^{n+1} + B x^n + C x^{n-1} + D x^{n-2}$ &c. then taking the increments,

$$\begin{aligned} \overline{n+1} \cdot A x^n &+ \frac{\overline{n+1} \cdot n}{2} A x^{n-1} x^2 + \frac{\overline{n+1} \cdot n \cdot \overline{n-1}}{2 \cdot 3} A x^{n-2} x^3 + \text{\&c.} \\ &+ n B x^{n-1} x + \frac{n \cdot \overline{n-1}}{2} B x^{n-2} x^2 + \text{\&c.} \\ &+ \overline{n-1} \cdot C x^{n-2} x + \text{\&c.} \\ &= a x^n + b x^{n-1} + c x^{n-2} + \text{\&c.} \end{aligned}$$

Then

Then by equating the terms, $A = \frac{a}{n+1 \cdot x} \cdot B = \frac{b}{n \cdot x} - \frac{n+1}{2} A x$,
 $C = \frac{c}{n-1 \cdot x} - \frac{n}{2} B x - \frac{n+1 \cdot n}{2 \cdot 3} A x^2$, $D = \frac{d}{n-1 \cdot x} -$
 $\frac{n-1}{2} C x - \frac{n \cdot n-1}{2 \cdot 3} B x^2 - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 3 \cdot 4} A x^3$, and so forward.

Then

The integral of $a x^n + b x^{n-1} + c x^{n-2} + d x^{n-3} + e x^{n-4} \&c.$

$$\begin{aligned}
 &= \left. \begin{aligned} &\frac{a}{n+1 \cdot x} x^{n+1} + \frac{b}{x \cdot x} x^n - \frac{n+1}{2} A x x^n \\ &+ \frac{c}{n+1 \cdot x} x^{n-1} - \frac{n}{2} B x x^{n-1} - \frac{n+1 \cdot n}{2 \cdot 3} A x^2 x^{n-1} \end{aligned} \right\} x^{n-1} \\
 &+ \left. \begin{aligned} &\frac{d}{n-2 \cdot x} x^{n-2} - \frac{n-1}{2} C x x^{n-2} - \frac{n \cdot n-1}{2 \cdot 3} B x^2 x^{n-2} \\ &- \frac{n+1 \cdot n \cdot n-1}{2 \cdot 3 \cdot 4} A x^3 x^{n-2} \end{aligned} \right\} x^{n-2} \\
 &+ \left. \begin{aligned} &\frac{e}{n-3 \cdot x} x^{n-3} - \frac{n-2}{2} D x x^{n-3} - \frac{n-1 \cdot n-2}{2 \cdot 3} C x^2 x^{n-3} \\ &- \frac{n \cdot n-1 \cdot n-2}{2 \cdot 3 \cdot 4} B x^3 x^{n-3} - \frac{n+1 \cdot n \cdot n-1 \cdot n-2}{2 \cdot 3 \cdot 4 \cdot 5} A x^4 x^{n-3} \end{aligned} \right\} x^{n-3} \\
 &+ \left. \begin{aligned} &\frac{f}{n-4 \cdot x} x^{n-4} - \frac{n-3}{2} E x x^{n-4} - \frac{n-2 \cdot n-3}{2 \cdot 3} D x^2 x^{n-4} \\ &- \frac{n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4} C x^3 x^{n-4} - \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4 \cdot 5} B x^4 x^{n-4} \\ &- \frac{n+1 \cdot n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} A x^5 x^{n-4} \end{aligned} \right\} x^{n-4} + \&c.
 \end{aligned}$$

Corol.

Corol. Integ. $x^n = \frac{x^{n+1}}{n+1} - \frac{x^n}{2} + \frac{n x}{12} x^{n-1} - \frac{n \cdot n-1 \cdot n-2 \cdot x^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^{n-3}$
 $+ \frac{n \dots n-4 \cdot x^5}{1 \dots 7 \cdot 6} x^{n-5} - \frac{n \dots n-6 \cdot x^7}{1 \dots 8 \cdot 5 \cdot 6} x^{n-7} +$
 $\frac{5 n \dots n-8 \cdot x^9}{1 \dots 10 \cdot 11 \cdot 6} x^{n-9} \quad \&c.$

EXAMPLE XI.

To find the integral of $\frac{a}{x x} + \frac{b}{x^3} + \frac{c}{x^4} + \frac{d}{x^5} \quad \&c.$ transf-
 form it by Prop. XII. thus,

$$\begin{aligned} \frac{a}{x x} &= \frac{a}{x x} + \frac{a x}{x \dots x_2} + \frac{2 a x^2}{x \dots x_3} + \frac{6 a x^3}{x \dots x_4} \quad \&c. \\ \frac{b}{x^3} &= \frac{b}{x \dots x_2} + \frac{3 b x}{x \dots x_3} + \frac{11 b x^2}{x \dots x_4} \\ \frac{c}{x^4} &= \frac{c}{x \dots x_3} + \frac{6 c x}{x \dots x_4} \\ \frac{d}{x^5} &= \frac{d}{x \dots x_4} \end{aligned}$$

$\&c.$ that is, $\frac{a}{x x} + \frac{b}{x^3} + \frac{c}{x^4} \quad \&c. = \frac{a}{x x} + \frac{a x + b}{x x x} +$
 $\frac{2 a x^2 + 3 b x + c}{x \dots x_3} + \frac{6 a x^3 + 11 b x^2 + 6 c x + d}{x \dots x_4} \quad \&c.$ Therefore the integral

of $\frac{a}{x x} + \frac{b}{x^3} + \frac{c}{x^4} + \frac{d}{x^5} + \frac{e}{x^6} \quad \&c. = - \frac{a}{x x} -$
 $\frac{a x + b}{2 x x x} - \frac{2 a x^2 + 3 b x + c}{3 x \dots x_2} - \frac{6 a x^3 + 11 b x^2 + 6 c x + d}{4 x x \dots x_3} -$

$\frac{24 a x^4 + 50 b x^3 + 35 c x^2 + 10 d x + e}{5 x x \dots x_4} \quad \&c.$ where the numbers are the

horizontal rows in Tab. II.

Otherwise,

Otherwise,

Let the integral be $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \&c.$ then taking the increments,

$$\begin{array}{rcll} -A \dot{x} x^{-2} & + A x^2 \dot{x}^{-3} & -A x^3 \dot{x}^{-4} & + A x^4 \dot{x}^{-5} \&c. \\ & -2B \dot{x} & + 3B \dot{x}^2 & -4B \dot{x}^3 \\ & & -3C \dot{x} & + 6C \dot{x}^2 \\ & & & -4D \dot{x} \\ & & & \&c. \end{array}$$

$$= a x^{-2} + b x^{-3} + c x^{-4} + d x^{-5}, \&c.$$

And equating the like coefficients, $A = -\frac{a}{x}$, $B = \frac{A x^2 - b}{2 x}$,
 $C = \frac{-A x^3 + 3B x^2 - c}{3 x}$, $D = \frac{A x^4 - 4B x^3 + 6C x^2 - d}{4 x}$, $\&c.$ Whence
 the integral of $\frac{a}{x x} + \frac{b}{x^3} + \frac{c}{x^4} + \frac{d}{x^5} + \frac{e}{x^6} \&c.$

$$\begin{aligned} \text{is} &= \frac{-a}{x x} \\ &+ \frac{A x^2 - b}{2 x x^2} \\ &+ \frac{-A x^3 + 3B x^2 - c}{3 x x^3} \\ &+ \frac{A x^4 - 4B x^3 + 6C x^2 - d}{4 x x^4} \\ &+ \frac{-A x^5 + 5B x^4 - 10C x^3 + 10D x^2 - e}{5 x x^5} \\ &+ \frac{A x^6 - 6B x^5 + 15C x^4 - 20D x^3 + 15E x^2 - f}{6 x x^6} \\ &+ \frac{-A x^7 + 7B x^6 - 21C x^5 + 35D x^4 - 35E x^3 + 21F x^2 - g}{7 x x^7} \end{aligned}$$

$\&c.$ where the numeral coefficients, are the same as those of the several powers of a binomial.

O

EXAMPLE

EXAMPLE XII.

To find the integral of $\frac{1}{z}$.

Let $Z = \text{hyp. log. of } z$, and $Z_1 = \text{hyp. log. of } \frac{z}{1}$ or of $z + \frac{z}{1} =$
 $Z + \frac{z}{z} - \frac{z^2}{2z^2} + \frac{z^3}{3z^3} - \frac{z^4}{4z^4} + \&c.$ by the nature of loga-
rithms. Then $Z_1 - Z$ or $Z = \frac{z}{z} - \frac{z^2}{2z^2} + \frac{z^3}{3z^3} - \frac{z^4}{4z^4} \&c.$
And $\frac{z}{z} = Z + \frac{z^2}{2z^2} - \frac{z^3}{3z^3} + \frac{z^4}{4z^4} \&c.$ And $\frac{1}{z} = \frac{Z}{z} +$
 $\frac{z}{2z^2} - \frac{z^2}{3z^3} + \frac{z^3}{4z^4} - \frac{z^4}{5z^5} \&c.$ Put $a = \frac{1}{2}z$, $b = -\frac{1}{3}z^2$,
 $c = \frac{1}{4}z^3$, $d = -\frac{1}{5}z^4$, $e = \frac{1}{6}z^5 \&c.$ And by the last example,
there will come out, the integral of $\frac{1}{z} = \frac{Z}{z} - \frac{1}{2z} - \frac{z}{12z^2} +$
 $\frac{z^3}{120z^4} - \frac{z^5}{252z^5} + \frac{z^7}{240z^8} \&c.$

Or thus,

Since $\frac{1}{z} = \frac{Z}{z} + \frac{z}{2z^2} - \frac{z^2}{3z^3} + \frac{z^3}{4z^4} \&c.$ These quantities
being transformed by Prop. XII. we have

$$\begin{aligned} \frac{Z}{z} &= \frac{Z}{z} \\ + \frac{z}{2z^2} &= + \frac{z}{2z \cdot z} + \frac{z^2}{2z \cdot z \cdot z} + \frac{z^3}{z \cdot z \cdot z} + \frac{3z^4}{z \cdot z \cdot z \cdot z} \&c. \\ - \frac{z^2}{3z^3} &= - \frac{z^2}{3z \cdot z \cdot z} - \frac{z^3}{z \cdot z \cdot z} - \frac{11z^4}{3z \cdot z \cdot z \cdot z} \\ + \frac{z^3}{4z^4} &= + \frac{z^3}{4z \cdot z \cdot z} + \frac{6z^4}{4z \cdot z \cdot z \cdot z} \\ - \frac{z^4}{5z^5} &= - \frac{z^4}{5z \cdot z \cdot z \cdot z} \\ &\&c. \end{aligned}$$

that

that is,

$$\frac{1}{z} = \frac{Z}{z} + \frac{z}{2z z_1} + \frac{z^2}{6z z_1 z_2} + \frac{z^3}{4z \dots z_3} + \frac{19z^4}{30z \dots z_4} \&c.$$

Therefore the integral of $\frac{1}{z} = \frac{Z}{z} - \frac{1}{2z} - \frac{z}{12z z_1} - \frac{z^2}{12z \dots z_2}$
 $- \frac{19z^3}{120z \dots z_3} - \frac{9z^4}{20z \dots z_4} - \frac{863z^5}{504z \dots z_5} - \frac{1375z^6}{168z \dots z_6} - \frac{33953z^7}{720z \dots z_7}$
 $\&c.$

E X A M P L E X I I I .

The integral of r^z is required, z being constant.

Put $s=r^z$, and the increment is $s-s_1 = r^z - r_1^z = r^{z+z_1} - r^z =$
 $\frac{r^z - 1}{r^z} \times r^z$; put the given quantity $r^z - 1 = R$, then $s-s_1$ or
 $s = R r^z$, and $\frac{s}{R} = r^z$, and taking the integral, $\frac{s}{R}$ or $\frac{r^z}{R} =$
 integ. of r^z .

Likewise $\frac{r^z}{RR} = \text{integ. of } \frac{r^z}{R}$, and $\frac{r^z}{R^3} = \text{int. of } \frac{r^z}{RR} \&c.$

By a like process the int. of $\frac{1}{r^z} = \frac{r^z}{r^z \times 1 - r^z}$.

Likewise,

To find the integral of vr^z , where z is constant.

According to Prop. IX. Let $V = v$, $X = r^z$, the rest as before;
 then since $[X] = \frac{X}{R} = \frac{r^z}{R}$, $[^2X] = \frac{r^z}{RR}$, $[^3X] = \frac{r^z}{R^3} \&c.$
therefore

therefore $[X] = \frac{r^z}{R}$, $[X]_1 = \frac{r^1}{RR}$, $[X]_2 = \frac{r^2}{R^3}$ &c.

Therefore we shall have

$$[vr^z] = \frac{vr^z}{R} - \frac{r^1}{RR} v + \frac{r^2}{R^3} v - \frac{r^3}{R^4} v + \&c.$$

where v may denote any compound quantity, whose increments may be found; and if any of the increments, v , v , v , &c. be 0, the series will terminate.

EXAMPLE XIV.

To find the integral of $r^z v \dots v$; z and v being constant.

Put $X = r^z$, $V = v \dots v$, $R = r^z - 1$; then by Ex. 13; $[X] = \frac{r^z}{R}$, $[X]_1 = \frac{r^1}{RR}$, $[X]_2 = \frac{r^2}{R^3}$, &c. and $V = \overline{n+1} \cdot v \dots v$, $V = \overline{n+1} \cdot n \cdot v \dots v$, $V = \overline{n+1} \cdot n \cdot \overline{n-1} \cdot v \dots v$, &c. Whence

$$\begin{aligned} \text{by Art. 3. the integral of } r^z v \dots v &= v \dots v \times \frac{r^z}{R} - \\ \overline{n+1} \times v \dots v \times \frac{r^1}{RR} &+ \overline{n+1} \cdot n \cdot v \dots v \times \frac{r^2}{R^3} - \\ \overline{n+1} \cdot n \cdot \overline{n-1} \times v \dots v \times \frac{r^3}{R^4} &\&c. = v \dots v \times \frac{r^z}{R} - \\ \overline{n+1} \cdot v \times \frac{r^z}{vR} A - n v \times \frac{r^z}{Rv} B &- \overline{n-1} \cdot v \times \frac{r^z}{R^2} C - \&c. \end{aligned}$$

where A, B, C, are the foregoing terms.

EXAMPLE

EXAMPLE XV.

To find the integral of $\frac{r^z}{v \dots v_n}$; z, v being constant.

Put $X = r^z$, $V = \frac{1}{v \dots v_n}$, $R = r^z - 1$, then by Ex. 13,

$$[X] = \frac{r^z}{R}, [X_1] = \frac{r^z}{RR}, [X_2] = \frac{r^z}{R^3} \&c. \text{ and } V = -\frac{\overline{n+1} \cdot v}{v \dots v_{n+1}}, V = \frac{\overline{n+1} \cdot \overline{n+2} \times v^2}{v \dots v_{n+2}}, V = -\frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \times v^3}{v \dots v_{n+3}}$$

$$\text{Therefore by Art. 3. int. of } \frac{r^z}{v \dots v_n} = \frac{r^z}{v \dots v_n R} + \frac{\overline{n+1} \times v r^z}{v \dots v_{n+1} R R} + \frac{\overline{n+1} \cdot \overline{n+2} \times v^2 r^z}{v \dots v_{n+2} R^3} + \frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3} \times v^3 r^3}{v \dots v_{n+3} R^4} \&c. =$$

$$\frac{r^z}{v \dots v_n R} + \frac{\overline{n+1} \cdot v r^z}{v R} A + \frac{\overline{n+2} \cdot v r^z}{v R} B + \frac{\overline{n+3} \cdot v r^z}{v R} C + \&c$$

where A, B, C, are the foregoing terms.

EXAMPLE XVI.

To find the integral of $r^z \times : a x^n + b x^{n-1} + c x^{n-2} + d x^{n-3} \&c.$

Let $r^z \times : A x^n + B x^{n-1} + C x^{n-2} + D x^{n-3} \&c.$ be the integral.

Put $x = r^z$, $v = A x^n + B x^{n-1} + C x^{n-2} \&c.$ Then $x = r^z - 1 \times r^z$, $x = r^z = r^z r^z$; then since the increment of

P

 $x v =$

$x v = v x + x v$, therefore finding the increment of v or $A x^n + B x^{n-1} \&c.$ and substituting the values of $v, \dot{v}, \ddot{x}, \dot{x}$, we have

$$\begin{aligned} & \ddot{r} - 1 \times \ddot{r} \times : A x^n + B x^{n-1} + C x^{n-2} \&c. \\ & + \ddot{r} \times \ddot{r} \times : v \\ & = \ddot{r} \times : a x^n + b x^{n-1} + c x^{n-2} \&c. \end{aligned}$$

that is, putting $R = \ddot{r} - 1$, $p = \ddot{r}$,

$$\begin{aligned} & R A x^n + R B x^{n-1} + R C x^{n-2} + R D x^{n-3} \&c. \\ & + p n A x + \frac{p n \cdot n-1 \cdot x^2}{2} A + \frac{p \cdot n \cdot n-1 \cdot n-2 \cdot x^3}{2 \cdot 3} A \\ & + p \cdot n-1 \cdot B x + \frac{p \cdot n-1 \cdot n-2}{2} B x^2 \\ & + p \cdot n-2 \cdot C x \end{aligned}$$

$$= a x^n + b + c + d \&c.$$

And by equating the terms $R A = a$, $R B + p n A x = b$, $R C + \frac{p n \cdot n-1 \cdot x^2}{2} A + p \cdot n-1 \cdot B x = c$, whence $A = \frac{a}{R}$, $B = \frac{b - p n A x}{R}$

$$C = \frac{c - \frac{1}{2} p n \cdot n-1 \cdot x^2 A - p \cdot n-1 \cdot B x}{R} \&c. \text{ and the integral of}$$

$$\ddot{r} \times : a x^n + b x^{n-1} + c x^{n-2} + d x^{n-3} \&c. = \ddot{r} \times \text{ into}$$

$$\left. \begin{aligned} & \frac{a x^n}{R} + \frac{b - p n A x}{R} \right\} x^{n-1} \left. \begin{aligned} & + c \\ & - p n \cdot \frac{n-1}{2} A x^2 \\ & - \frac{p \cdot n-1 \cdot B x}{R} \end{aligned} \right\} x^{n-2}$$

(D)

(E)

$$\left. \begin{aligned} & + d \\ & - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} p A x^3 \\ & - \frac{n-1}{1} \cdot \frac{n-2}{2} p B x^2 \\ & - \frac{n-2}{1} \cdot p C x \end{aligned} \right\} x^{n-3} \left. \begin{aligned} & + e \\ & - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} p A x^4 \\ & - \frac{n-1}{1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} p B x^3 \\ & - \frac{n-2}{1} \cdot \frac{n-3}{2} p C x^2 \\ & - \frac{n-3}{1} \cdot p D x \end{aligned} \right\} x^{n-4} \&c.$$

where $x, \dot{x}$ are constant.

EXAMPLE

EXAMPLE XVII.

To find the integral of $r^z \times : \frac{a}{x} + \frac{b}{x x} + \frac{c}{x^3} + \frac{d}{x^4} \&c.$
 where x, z are constant.

Let the integral be $r^z \times : \frac{A}{x} + \frac{B}{x x} + \frac{C}{x^3} + \frac{D}{x^4} \&c.$
 Then putting $x = r^z$, $v = \frac{A}{x} + \frac{B}{x x} \&c.$ $R = r^z - 1$, $p = r^z$,
 and computing the value of $v x + x v$, as in the last, we shall have

$$\begin{aligned} & \frac{RA}{x} + \frac{RB}{x x} + \frac{RC}{x^3} + \frac{RD}{x^4} \&c. \\ & - \frac{p A x}{x x} + \frac{p A x^2}{x^3} - \frac{p A x^3}{x^4} \\ & - \frac{2 p B x}{x^3} + \frac{3 p B x^2}{x^4} \\ & - \frac{3 p C x}{x^4} \\ & = \frac{a}{x} + \frac{b}{x x} + \frac{c}{x^3} + \frac{d}{x^4} \&c. \end{aligned}$$

Then $RA = a$, $RB - pAx = b$, $RC + pAx^2 - 2pBx = c$, $\&c.$
 whence $A = \frac{a}{R}$, $B = \frac{b + pAx}{R}$, $C = \frac{c - pAx^2 + 2pBx}{R}$,
 $D = \frac{d + pAx^3 - 3pBx^2 + 3pCx}{R}$, $\&c.$ Whence the integral
 of $r^z : x \frac{a}{x} + \frac{b}{x x} + \frac{c}{x^3} + \frac{d}{x^4} \&c. = r^z \times \frac{A}{x} + \frac{B}{x x}$
 $+ \frac{C}{x^3} + \frac{D}{x^4} \&c.$ where the numeral coefficients of A, B, C , $\&c.$
 are those of the powers of a binomial.

EXAMPLE

EXAMPLE XVIII.

To find the integral of $r^z \times : a + b x + c x x + d x . . x + e x . . . x$
&c. z and x being constant.

Let the integral be $r^z \times : A + B x + C x x + D x x x$ &c. ($= xv$)
then taking the increments and dividing by r^z , we have

$$\begin{aligned} & RA + RBx + RCxx + RDx . . x (= vx) \\ & + pBx + 2pCxx + 3pDx x x + 4pEx . . x x (= xv) \\ & = a + bx + cxx + dx . . x \end{aligned}$$

And by equating the like terms, $RA + pBx = a$, $RB + 2pCx = b$, &c.
where A is indeterminated; then $B = \frac{a - RA}{px}$, $C = \frac{b - RB}{2px}$, $D = \frac{c - RC}{3px}$

&c. and the integral of $r^z \times : a + bx + cxx + dx . . x$ &c. =

$$r^z \times : A + \frac{a - RA}{px} x + \frac{b - RB}{2px} xx + \frac{c - RC}{3px} xxx \text{ &c. where}$$

$R = r^z - 1$, $p = r^z$, B, C, D , the preceeding terms.

EXAMPLE XIX.

To find the integral of $r^z : x a + \frac{b}{x} + \frac{c}{xx} + \frac{d}{xxx} + \frac{e}{xxxx}$ &c.

Assume $r^z \times : A + \frac{B}{x} + \frac{C}{xx} + \frac{D}{xxx} + \frac{E}{xxxx} + \text{&c. for}$

it; then taking the increment, and dividing by r^z , we have

$$RA +$$

$$\begin{aligned}
 & RA + \frac{RB}{x} + \frac{RC}{x \cdot x_1} + \frac{RD}{x \cdot x \cdot x_2} + \frac{RE}{x \cdot x \cdot x_3} \&c. \\
 & \quad - \frac{p B x}{x \cdot x_1} - \frac{2 p C x}{x \cdot x \cdot x_2} - \frac{3 p D x}{x \cdot x \cdot x_3} \\
 & = a + \frac{b}{x} + \frac{c}{x \cdot x_1} + \frac{d}{x \cdot x \cdot x_2} + \frac{e}{x \cdot x \cdot x_3} \&c.
 \end{aligned}$$

Whence $RA = a$, $RB = b$, $RC = c + pBx$, $RD = d + 2pCx$, $\&c.$
 whence the integral of $r^z \times : a + \frac{b}{x} + \frac{c}{x \cdot x_1} + \frac{d}{x \cdot x \cdot x_2} + \frac{e}{x \cdot x \cdot x_3}$

$$\&c. = \frac{r^z}{R} \times : a + \frac{b}{x} + \frac{c + pBx}{x \cdot x_1} + \frac{d + 2pCx}{x \cdot x \cdot x_2} + \frac{e + 3pDx}{x \cdot x \cdot x_3} \&c.$$

where $R = r^z - 1$, $p = r^z$ as before.

Or expunging B, C, D, $\&c.$ and putting P, Q, S, for the whole preceding terms; then int. $r^z \times$ into $a + \frac{b}{x} + \frac{c}{x \cdot x_1} + \frac{d}{x \cdot x \cdot x_2} \&c.$

$$\begin{aligned}
 & = \frac{r^z}{R} \times \text{into } a + \frac{b}{x} + \frac{x r^z}{R x_1} P \left\{ + \frac{2 x r^z}{R x_2} Q \right\} \\
 & \quad \left[+ \frac{c}{x \cdot x_1} \right] \left\{ + \frac{d}{x \cdot x \cdot x_2} \right\} \\
 & \quad \left[+ \frac{3 x r^z}{R x_3} S \right] \left\{ + \frac{4 x r^z}{R x_4} T \right\} \&c. \\
 & \quad \left[+ \frac{e}{x \cdot x \cdot x_3} \right] \left\{ + \frac{f}{x \cdot x \cdot x_4} \right\}
 \end{aligned}$$

Q

EXAMPLE

EXAMPLE XX.

To find the integral of $\frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v}$.

Here $\overline{v-r} \times \overline{v-r} = v \cdot v - r \cdot v - r \cdot v + r \cdot r = v \cdot v - 2rv + rr$,

hence $\frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v} = 1 - \frac{2r}{v} + \frac{r-v}{v \cdot v} \cdot r$. Therefore integral of

$\frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v} = \frac{v}{v} - 2r \left[\frac{1}{v} \right] - \frac{r-v}{v \cdot v} \cdot r$. And $\left[\frac{1}{v} \right]$ is had

by Example 12.

EXAMPLE XXI.

To find the integral of $\frac{\overline{v-r} \cdot \overline{v-r} \cdot \overline{v-r}}{v \cdot v \cdot v}$.

Here we have $\overline{v-r} \cdot \overline{v-r} \cdot \overline{v-r} = v \cdot v \cdot v - 2rrr + r \cdot r \cdot r - r \cdot v \cdot v \times \overline{v-r}$
 $= v \cdot v \cdot v - 2rrv + rr^2 - r \cdot v \cdot v \times \overline{v-r} + 2r^2v - r^3$, but —
 $2rrv + 2rrr = -2rv \times \overline{v-r} + 2rrv = -2rvv + 2rr - 2rv \times v$
 $= -2rvv + 2rr - 2rv \times v = -2rvv + 2rr \left\{ \frac{v}{v} - 2v \right\} + 4rv^2 - 4rrv$;

therefore $\frac{\overline{v-r} \cdot \overline{v-r} \cdot \overline{v-r}}{v \cdot v \cdot v} = 1 - \frac{3r}{v} + \frac{r-v}{v \cdot v} \times 3r +$

$\frac{4rv^2 - 5rrv - r^3}{v \cdot v \cdot v}$. Therefore the integral of $\frac{\overline{v-r} \cdot \overline{v-r} \cdot \overline{v-r}}{v \cdot v \cdot v}$

$= \frac{v}{v} - 3r \left[\frac{1}{v} \right] + \frac{r-v}{v \cdot v} \times 3r + \frac{4rv^2 - 5rrv - r^3}{2v \cdot v \cdot v}$.

And the like for more quantities.

EXAMPLE

EXAMPLE XXII.

To find the integral of $\frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v} \times a^z$, where v and r are constant.

$$\begin{aligned} \text{Here } \frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v} &= \frac{v \cdot v - r \times \overline{v+v} + r r}{v \cdot v} = \frac{v \cdot v - r \times \overline{v+v} - v + r r}{v \cdot v} \\ &= 1 - \frac{2r}{v} + \frac{r r + r v}{v \cdot v} \end{aligned}$$

Then by Examp. 19. the integral of $\frac{\overline{v-r} \cdot \overline{v-r}}{v \cdot v} \times a^z =$

$$\frac{a^z}{R} \times \text{into } 1 - \frac{2r}{v} + \frac{v \cdot a^z}{R \cdot v} A \left\{ + \frac{2 v \cdot a^z}{R \cdot v} B + \frac{3 v \cdot a^z}{R \cdot v} C + \&c, \right.$$

$$\left. + \frac{r r + r v}{v \cdot v} \right\}$$

where $R = a^z - 1$.

After the same manner is found the integral of $\frac{\overline{v-r}}{v} a^z$, of $\frac{\overline{v-r} \cdot \overline{v-r} \cdot \overline{v-r}}{v \cdot v \cdot v} a^z$, &c.

EXAMPLE XXIII.

To find the integral of $\frac{n}{z} s = s = 0$, or $\frac{z+n}{z} s = s$, where n is given, and z increases uniformly.

Here $s = \frac{z}{n} s$. And by Art. 3. the int. of $s \cdot z$ is $s z - s z$,
therefore

therefore the integral of $s = \frac{z}{n} s$ is $s = \frac{1}{n} \times \overline{zs} - \overline{s z} = \frac{zs}{n} - \frac{s z}{n}$. Therefore $ns = \overline{zs} - \overline{s z} = \overline{zs} - s z - \overline{zs}$, whence $\overline{z+n} \times s = \overline{z-z} \times s$; and $s = \frac{z-z}{z+n} s$.

EXAMPLE XXIV.

To find the integral of $\frac{z-n}{rz} s - s = s$, where n, r are constant, and may be either affirmative or negative.

Suppose the integral, $s = vs$; then taking the increment, $s = vs + v s$, put $r-1 = p$. Then $\overline{1-v} \times s = v s$, and $s = \frac{vs}{1-v} = \frac{-rzs}{pz+n}$ by the given equation. And $\overline{pz+n} \times v = -rz \times \overline{1-v}$, and $pv + \frac{nv}{z} + r - rv = 0$.

$$\text{Assume } v = a + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3} \&c.$$

$$\text{Then } v = \frac{-bz}{z^2} - \frac{2cz}{z^3} - \frac{3d}{z^4} \&c.$$

$$v = a + \frac{b}{z} + \frac{c}{z^2} + \frac{d}{z^3} \&c.$$

$$\text{Or } v = a + \frac{b}{z} + \frac{c-bz}{z^2} + \frac{d-2cz}{z^3} \&c.$$

Then,

Then,

$$\begin{array}{l|l}
 p v_1 & p a + \frac{p b}{z} + \frac{p c - p b z}{z z_1} + \frac{p d - 2 p c z}{z z z_1 z_2} \&c. \\
 \frac{n v_1}{z} & + \frac{n a}{z} + \frac{n b}{z z_1} + \frac{n c}{z z z_1 z_2} \&c. \\
 r - r v_1 & r * + \frac{r b z}{z z_1} + \frac{2 r c z}{z z z_1 z_2} \&c.
 \end{array}$$

= 0.

Then equating the homologous terms. $p a - r = 0$, and $a = \frac{-r}{p}$.
 $p b + n a = 0$, and $b = \frac{-n a}{p} = \frac{n r}{p p}$. $p c - p b z + n b + r b z = 0$,
 and $c = \frac{p z - r z - n b}{p} = -\frac{n + z}{p} b = -\frac{n + z}{p} \times \frac{n r}{p p}$. Likewise
 $d = \frac{2 p - 2 r \cdot z - n c}{p} = -\frac{n + 2 z}{p} \times -\frac{n + z}{p} \times \frac{n r}{p p} \&c.$ There-
 fore $s = s \times : \frac{-r}{p} + \frac{n r}{p p z} - \frac{n + z}{p z z_1} \times \frac{n r}{p p} - \frac{n + 2 z}{p z z_1 z_2} \times -\frac{n + z}{p}$
 $\times \frac{n r}{p p} \&c.$ that is $s = \frac{-r s}{p} - \frac{n}{p z} A - \frac{n + z}{p z_1} B - \frac{n + 2 z}{p z_2} C -$
 $\frac{n + 3 z}{p z_3} D \&c.$

SCHOLIUM.

When $r=1$, this Example becomes the same as Ex. 23.

EXAMPLE XXV.

To find the integral of $\frac{z-m \cdot z-n}{z z_1} s = s.$

This reduced is $\overline{m+n+z} \times s - \frac{mn}{z} s + z s = 0$, put $a = m+n+z$,
 $b = -mn$, then $a s + \frac{b}{z} s + z s = 0.$

R

Suppose

Suppose as before $s = v s$, whence we get $s = \frac{v s}{1-v} = \frac{-z s}{a + \frac{b}{z}}$,

whence $a v + \frac{b}{z} v + z \times \frac{1}{1-v} = 0$.

Assume $v = a z + \beta + \frac{\gamma}{z} + \frac{\delta}{z z} + \frac{\epsilon}{z z z} \&c.$

Then $v = a z + \beta + \frac{\gamma}{z} + \frac{\delta}{z z} + \frac{\epsilon}{z z z} \&c.$

Or $v = a z + \beta + \frac{\gamma}{z} + \frac{\delta - \gamma z}{z z} + \frac{\epsilon - 2\delta z}{z z z} + \frac{\zeta - 3\epsilon z}{z \dots z} \&c.$

$v = a z + \frac{\gamma z}{z z} + \frac{2\delta z}{z z z} \&c.$

$1-v = 1 - a z + \frac{\gamma z}{z z} + \frac{2\delta z}{z z z} + \frac{3\epsilon z}{z \dots z} \&c.$

$z \times \frac{1}{1-v} = \frac{1}{1-a z} \times z + \frac{\gamma z}{z z} + \frac{2\delta z \times z - z}{z z z} + \frac{3\epsilon z \times z - 2z}{z \dots z}$
 $+ \frac{4\zeta z \times z - 3z}{z \dots z} + \frac{5\eta z \times z - 4z}{z \dots z} + \frac{6\theta z \times z - 5z}{z \dots z}$
 $\&c.$

$\&c.$

Whence

$$\begin{array}{l|l}
 a v + \frac{b}{z} v + z \times \frac{1}{1-v} & a a z + a \beta + \frac{a \gamma}{z} + \frac{a \delta - a \gamma z}{z z} + \frac{a \epsilon - 2 \delta a z}{z z z} \&c. \\
 & + b a + \frac{b a z + b \beta}{z} + \frac{b \gamma}{z z} + \frac{b \delta}{z z z} \&c. \\
 & \frac{1}{1-a z} \times z + \frac{\gamma z}{z z} + \frac{2 \delta z}{z z z} + \frac{3 \epsilon z - 1.2 \delta z^2}{z z z} \&c. \\
 & = 0.
 \end{array}$$

Then

Then

$$\begin{aligned} a\alpha + 1 - \alpha z &= 0, \quad a\beta + b\alpha = 0, \quad a\gamma + b\alpha z + b\beta + \gamma z = 0, \\ a\delta - a\gamma z + b\gamma + 2\delta z &= 0, \quad a\varepsilon - 2\delta az + b\delta + 3\varepsilon z - 1.2\delta z^2 = 0, \\ a\zeta - 3a\varepsilon z + b\varepsilon + 4\zeta z - 2.3\varepsilon z &= 0, \quad a\eta - 4a\zeta z + b\zeta + 5\eta z - 3.4\zeta z^2 = 0, \\ a\vartheta - 5a\eta z + b\eta + 6\vartheta z - 4.5\eta z^2 &= 0, \quad \&c. \text{ whence} \end{aligned}$$

$$\alpha = \frac{-1}{a-z}, \quad \beta = \frac{-b}{a}, \quad \gamma = \frac{az-b}{a+z}, \quad \delta = \frac{az-b}{a+2z} \gamma.$$

$$\varepsilon = \frac{2az + 1.2z^2 - b}{a+3z} \delta, \quad \zeta = \frac{3az + 2.3z^2 - b}{a+4z} \varepsilon, \quad \eta = \frac{4az + 3.4z^2 - b}{a+5z} \zeta.$$

$$\vartheta = \frac{5az + 4.5z^2 - b}{a+6z} \eta, \quad \&c. \text{ Consequently}$$

$$\begin{aligned} s &= s \times \text{into} \frac{-z}{a-z} + \frac{b}{a \times a-z} + \frac{az-b}{z \times a+z} \times \frac{b}{a \times a+z} + \\ &\frac{az-b}{z \times a+2z} \times \frac{az-b}{a+z} \times \frac{b}{a \times a+z} \quad \&c. \text{ that is,} \end{aligned}$$

$$\begin{aligned} s &= \frac{-zs}{a-z} + \frac{bs}{a \times a-z} + \frac{az-b}{z \times a+z} B + \frac{az-b}{z \times a+2z} C + \\ &\frac{2az + 1.2z^2 - b}{z \times a+3z} D + \frac{3az + 2.3z^2 - b}{z \times a+4z} E + \frac{4az + 3.4z^2 - b}{z \times a+5z} F + \\ &\frac{5az + 4.5z^2 - b}{z \times a+6z} G + \&c. \end{aligned}$$

EXAMPLE XXVI.

To find the integral of $\frac{\overline{z-m} \times \overline{z-n}}{z \times \overline{z-n}} s = s.$

$$\begin{aligned} \text{Let } s &= v s \times \overline{z-n}. \text{ Then } s = \frac{s v \times \overline{z-n}}{1} - \frac{s v \times \overline{z-n}}{1} = \\ s + s \times \frac{v \times \overline{z-n}}{1} - s v \times \overline{z-n}, \text{ or } 1 + \frac{v \times \overline{z-n}}{1} - \frac{v \times \overline{z-n}}{1} \times s &= \frac{v \times \overline{z-n}}{1} \times s; \end{aligned}$$

$$\text{whence } \frac{1 + \frac{v \times \overline{z-n}}{1}}{\frac{v \times \overline{z-n}}{1}} - s = s = \frac{\overline{z-m} \times \overline{z-n}}{z \times \overline{z-n}} s - s, \text{ there-}$$

fore

fore $\frac{1 + v \times z - n}{v} = \frac{z - m \times z - n}{z}$, or $\frac{1}{z - n} + v = \frac{z - m}{z} v$;

and $v - \frac{m v}{z} - v = \frac{1}{z - n}$, that is, $\frac{m v}{z} - v + \frac{1}{z - n} = 0$.

Assume $v = A + \frac{B}{z} + \frac{C}{z z} + \frac{D}{z z z} \&c.$

Then $v = -\frac{B z}{z z} - \frac{2 C z}{z z z} - \frac{3 D z}{z \dots z} \&c.$

$v = A + \frac{B}{z} + \frac{C}{z z} + \frac{D}{z \dots z} \&c.$

And $\frac{1}{z - n} = \frac{1}{z} + \frac{n}{z z} + \frac{n n}{z^3} + \frac{n^3}{z^4} \&c. = (\text{by Ex. 19. Pr. XII.})$

$\frac{1}{z} + \frac{n}{z z} + \frac{n \cdot n + z}{z \dots z} + \frac{n \cdot n + z \cdot n + 2 z}{z \dots z} \&c. \text{ put } n, n, n,$

for $n + z$, $n + 2z$, $n + 3z$ &c. Then

$$\left. \begin{array}{l} + \frac{m v}{z} \left| \begin{array}{l} \frac{m A}{z} + \frac{m B}{z z} + \frac{m C}{z z z} + \frac{m D}{z \dots z} \&c. \\ + \frac{B z}{z z} + \frac{2 C z}{z z z} + \frac{3 D z}{z \dots z} \&c. \\ + \frac{1}{z - n} \left| \begin{array}{l} \frac{1}{z} + \frac{n}{z z} + \frac{n n}{z z z} + \frac{n n n}{z \dots z} \&c. \end{array} \right. \end{array} \right\} = 0.$$

Then equating the coefficients, $m A + 1 = 0$, $m B + B z + n = 0$, $m C + 2 C z + n n = 0$, $m D + 3 D z + n n n = 0$, &c. whence $A = -$

$\frac{1}{m}$, $B = \frac{-n}{m + z}$, $C = \frac{-n n}{m + 2z}$, $D = -\frac{n n n}{m + 3z} \&c. \text{ there-}$

fore $s = s \times z - n \times \text{into } -\frac{1}{m} - \frac{n}{m + z \cdot z} - \frac{n n}{m + 2z \cdot z z} \&c.$

that

that is, $s = \overline{z-n} \times s$ into $\frac{-1}{m} + \frac{\frac{n}{z} P}{m+z} + \frac{\frac{\frac{n}{z}}{z} Q}{m+2z} + \frac{\frac{\frac{\frac{n}{z}}{z}}{z} R}{m+3z} +$
 $\frac{\frac{\frac{\frac{n}{z}}{z}}{z} S}{m+4z} + \frac{\frac{\frac{\frac{\frac{n}{z}}{z}}{z}}{z} T}{m+5z} + \&c.$ where P, Q, R, &c. are the preceeding
 numerators with their figs.

EXAMPLE XXVII.

To find the integral of $\frac{\overline{z-m} \cdot \overline{z-n}}{z \cdot \overline{z-n}} r s = s.$

Let $s = \overline{z-n} \cdot v s$, then we shall have, as in the last,

$$\frac{1 + v \cdot \overline{z-n}}{v \cdot \overline{z-n}} = \frac{\overline{z-m} \cdot \overline{z-n}}{z \cdot \overline{z-n}} r, \text{ and } \frac{1}{z-n} + v = \frac{1}{v} r \times \frac{z-m}{z}, \text{ or}$$

$$\frac{1}{z-n} + v - r \frac{1}{v} + \frac{m r v}{z} = 0, \text{ that is, } \frac{1}{1-r} \cdot \frac{1}{v} + \frac{m r}{z} \frac{1}{v} - \frac{1}{v} +$$

$$\frac{1}{z-n} = 0, \text{ put } 1-r = p. \text{ And } v = \frac{A}{z} + \frac{B}{z z_1} + \frac{C}{z z_1 z_2} + \frac{D}{z \dots z_3}$$

&c.

Then,

+ p v	$\frac{p A}{z} + \frac{p B - p A z}{z z_1} + \frac{p C - 2 p B z}{z z_1 z_2} + \frac{p D - 3 p C z}{z \dots z_3} \&c.$
+ \frac{m r}{z} v	$+ \frac{m r A}{z z_1} + \frac{m r B}{z z_1 z_2} + \frac{m r C}{z \dots z_3} \&c.$
- v	$+ \frac{A z}{z z_1} + \frac{2 B z}{z z_1 z_2} + \frac{3 C z}{z \dots z_3} \&c.$
\frac{1}{z-n}	$\frac{1}{z} + \frac{n}{z z_1} + \frac{n n}{z z_1 z_2} + \frac{n n n}{z \dots z_3} \&c.$
= 0.	S

Put

Put $n_1, n_2, \&c.$ for $n+z, n+2z, \&c.$ and $m_1, m_2, \&c.$ for $m+z, m+2z, \&c.$ then equating the coefficients, $pA+1=0$, $pB-pAz+mrA+Az+n=0$, $pC-2pBz+mrB+2Bz+n_1=0$, $\&c.$ whence $A = \frac{-1}{1-r}$,
 $B = -\frac{mrA+n}{1-r}$, $C = -\frac{mrB+n_1}{1-r}$, $D = -\frac{mrC+n_2}{1-r}$ $\&c.$

Whence

$$s = \frac{-1}{1-r} \cdot s \times \text{into } \frac{-1}{1-r \cdot z} - \frac{mrA+n}{1-r \cdot z \cdot z} - \frac{mrB+n_1}{1-r \cdot z \cdot z \cdot z} - \frac{mrC+n_2}{1-r \cdot z \cdot z \cdot z \cdot z} - \frac{mrD+n_3}{1-r \cdot z \cdot z \cdot z \cdot z \cdot z}, \&c. \text{ where } A, B, C, \&c. \text{ are}$$

the preceding terms with their signs, without the quantities $z, z \cdot z, z \cdot z \cdot z, \&c.$

EXAMPLE XXVIII.

Let $\frac{zx+mz+n}{zx+pz} s = s = s + s$, to find the integral.

By reducing the equation, $\overline{zx+mz+n} \times s = \overline{zx+pz} \times s + \overline{zx+pz} \times s$; and $\overline{m-p} \cdot z + n : \times s = \overline{zx+pz} : \times s$; put $m-p=q$, then dividing by z , $q \cdot s + \frac{n}{z} s = \overline{z+p} \times s$; and $s = \frac{z+p}{q+\frac{n}{z}} s$.

Let $s = v s$, then $s = v s + v s$, and $s = \frac{v s}{1-v} = \frac{z+p}{q+\frac{n}{z}} s$, as

found before; therefore $\frac{v}{1-v} = \frac{z+p}{q+\frac{n}{z}}$; which reduced, gives

$$q \cdot v + \frac{n v}{z} - \overline{z+p} \times \overline{1-v} = 0,$$

Assume

$$\text{Assume } v = Az + B + \frac{C}{z} + \frac{D}{z^2} + \frac{E}{z^3} \&c.$$

$$\text{Then } v = Az - \frac{Cz}{z^2} - \frac{2Dz}{z^3} - \frac{3Ez}{z^4} \&c.$$

$$1-v = +Az + \frac{Cz}{z^2} + \frac{2Dz}{z^3} + \frac{3Ez}{z^4} \&c.$$

$$v = Az + B + \frac{C}{z} + \frac{D}{z^2} + \frac{E}{z^3} \&c.$$

$$\text{Or } v = Az + B + \frac{C}{z} + \frac{D-Cz}{z^2} + \frac{E-2Dz}{z^3} \&c.$$

Whence we shall have

$$\begin{array}{l} +qv \left\{ \begin{array}{l} qAz + B \\ + Az \end{array} \right\} q + \frac{qC}{z} + \frac{qD-qCz}{z^2} + \frac{qE-2qDz}{z^3} \&c. \\ + \frac{nv}{z} * + nA + \frac{nB+nAz}{z} + \frac{nC}{z^2} + \frac{nD}{z^3} \&c. \\ -z \times 1-v \left\{ \begin{array}{l} -z \\ + Az \end{array} \right\} * - \frac{Cz}{z} - \frac{2Dz-Cz^2}{z^2} - \frac{3Ez-4Dz^2}{z^3} \&c. \\ -p \times 1-v \left\{ \begin{array}{l} -p \\ + pA \end{array} \right\} * - \frac{pCz}{z^2} - \frac{2pDz}{z^3} \&c. \end{array}$$

And equating the coefficients, $qA-1+Az=0$, $Bq+Agz+nA-p+pAz=0$, $qC+Bn+nAz-Cz=0$, $\&c.$ whence $A = \frac{1}{q+z}$,

$$B = -\frac{n+pz+qz \times A-p}{q} = \frac{n+mz \times A-p}{q}, \quad C = -\frac{zA+B}{q-z},$$

$$D = -\frac{n+z^2-pz-qz}{q-2z} \quad C = -\frac{n+z^2-mz}{q-2z} \quad C,$$

$$E = -\frac{n+4z^2-2pz-2qz}{q-3z} \quad D = -\frac{n+4z^2-2mz}{q-3z} \quad D,$$

$$F = -\frac{n+9z^2-3mz}{q-4z} \quad E, \quad G = -\frac{n+16z^2-4mz}{q-5z} \quad F, \&c.$$

whence

whence $s = s \times$ into $Az + B + \frac{C}{z} + \frac{D}{z^2} + \frac{E}{z^3} + \frac{F}{z^4} + \frac{G}{z^5} \&c.$ But this series will fail when any denominator happens to be 0.

After the same manner the integrals of more compound quantities, made up of s and s , may be found in terms of s and z ; but then the more compounded they are, the more compounded the series will come out.

EXAMPLE XXIX.

Let $ms + ns + ps + qs \&c. = 0$, to find the integral s , expressed by s, s, s ; or by s, s, s , &c.

The integral of the given equation is $ms + ns + ps + qs \&c. = 0$; but $ns = ns + ns$, by the notation; and $ps = ps + ps = ps + ps + ps$. Also $qs = qs + qs = qs + qs + qs = qs + qs + qs + qs \&c.$

$$\begin{array}{l} \text{Whence} \\ ms \\ + ns + ns \\ + ps + ps + ps \\ + qs + qs + qs + qs \end{array} \left. \vphantom{\begin{array}{l} ms \\ + ns + ns \\ + ps + ps + ps \\ + qs + qs + qs + qs \end{array}} \right\} = 0.$$

$$\text{Therefore } s = - \frac{\overline{n+p+q} \&c. \times s + \overline{p+q} \times s + q \times s \&c.}{m+n+p+q \&c.}$$

$$\text{Or } s = - \frac{ns + p \times \overline{s+s} + q \times \overline{s+s+s} \&c.}{m+n+p+q}$$

Or

Or thus,

$$\begin{array}{l} \text{By Prop. II. } ms = ms \\ n s = ns + n s \\ p s = ps + 2 p s + p s \\ q s = qs + 3 q s + 3 q s + q s \\ \text{\&c.} \end{array} \left. \vphantom{\begin{array}{l} ms \\ ns \\ ps \\ qs \end{array}} \right\} = 0.$$

$$\text{Whence } s = - \frac{\overline{n+2p+3q} \text{ \&c. } \times s + \overline{p+3q} \times s + q s}{m+n+p+q \text{ \&c.}}$$

A Table of Increments and their Integrals:

Forms.	Increments.	Integrals.
1	a , given	$\frac{a X}{X}, \frac{a Y}{Y}, \frac{a Z}{Z}, \&c.$
2	Z when constant not constant	Z , or $Z_1, Z_2, Z_3, Z_4, \&c.$ Z only.
3	$\frac{Z}{Z}$, or $\frac{Z}{Z_1}, \frac{Z}{Z_2}, \frac{Z}{Z_3}, \frac{Z}{Z_4},$ &c. Z constant.	infinite. or see Ex. 12. Prop. XIII.
4	$\frac{Z}{Z_1} \frac{Z}{Z_2} \frac{Z}{Z_3} \frac{Z}{Z_4}$ Z constant	$\frac{Z_3 Z_2 Z_1 Z}{5Z}$ $\frac{Z_1 Z_2 Z_3 Z_4}{5Z}$
5	$\frac{1}{Z_1 Z_2 Z_3 Z_4}$ Z constant	$-\frac{1}{3 Z_1 \times Z_2 Z_3 Z_4}$ $-\frac{1}{3 Z_1 \times Z_2 Z_3 Z_4}$
6	$ZX + XZ$	XZ .
7	$\frac{ZX - XZ}{Z}$	$\frac{X}{Z}$
8	$ZX - XZ = 0.$	$\frac{X}{Z} = A.$
9	$ZX + r X Z = 0.$ Z given.	$X Z \dots Z = A.$ $r-1$

PROP. XIV.

To correct any integral found by the foregoing rules.

WHEN an integral has been obtained by any of the former methods, it is very often imperfect, till it be corrected by the following

RULE.

1. Instead of each several variable quantity in the integral, substitute such a determinate value thereof, as it is known to have in some particular case; then subtract each side of the resulting equation from the correspondent side of the integral; and the remaining equation is the correct integral required.
2. Or add some constant quantity to one side of the equation; which may afterwards be determined, by substituting the several values of the variable quantities, for any particular known case.
3. Any particular part of the integral may also be found, by substituting the value of the variable quantity, in two several known cases; for then the difference of the equations gives the corresponding part of the integral.

EXAMPLE I.

In the equation $v = \frac{x^x}{z}$. Let $x = 0$, when $v = 0$, then
 $v - 0 = \frac{x^x}{z} - 0$, that is $v = \frac{x^x}{z}$, as at first; and therefore it needs no correction.

EXAMPLE II.

In the equation $v v = \frac{x^x x}{2.3}$. Let v be $= 0$, when $x = 4$, x being $= 1$, the equation becomes $1 \times 0 = \frac{4.3.2}{2.3} = 4$.

Then

$$\text{Then } v \ v = \frac{\overset{x}{-1} \overset{x}{-2}}{2.3}$$

$$\text{Subfr. } \frac{0 \dots 4}{\overset{x}{-1} \overset{x}{-2}}$$

$$\text{Whence } v \ v = \frac{\overset{x}{-1} \overset{x}{-2}}{2.3} - 4.$$

EXAMPLE III.

Let $z = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12}$, and let $x = 2$, when $z = 1$, and $x = 1$, constant; by substitution the equation becomes $1 = \frac{3.4.5.6}{12} = 30$.

$$\text{Then } z = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12}$$

$$1 \dots 30$$

then $z - 1 = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12} - 30$, or $z = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12} - 29$, the correct integral.

Or thus,

Let $z = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12} + A$. Now when $z = 1$, and $x = 2$, we have

$$1 = \frac{3.4.5.6}{12} + A, \text{ and } A = 1 - 30 = -29; \text{ whence}$$

$$z = \frac{\overset{x}{1} \overset{x}{2} \overset{x}{3} \overset{x}{4}}{12} - 29.$$

EXAMPLE IV.

Let $z = 2x$, and when $x = 1$, let $z = 3$, $x = 1$, then $2x = 2$. Then

$$z = 2x$$

$$3 \dots 2$$

$$z - 3 = 2x - 2,$$

or $z = 2x + 1$, the correct integral.

U

EXAMPLE

EXAMPLE V.

Suppose $s = a + \frac{b}{z} - \frac{c}{z^2}$, where $z = 2$.

And let s be $= p$ and q , when $z = 10$ and 100 respectively; then
 $p - q = a + \frac{b}{10} - \frac{c}{10 \cdot 12} - a - \frac{b}{100} + \frac{c}{100 \cdot 102}$ or $p - q =$
 $\frac{9}{100}b - \frac{7}{850}c.$

PROP. XV.

To resolve a problem by the Method of Increments.

RULES.

1. **W**HEN there is proposed a series of quantities, whose construction and progression is known, and their relation to one another given; and when the differences, that is, the increments of the several variable quantities concerned therein, can be computed and expressed by equations; and it is required from these increments to find the quantities themselves; then, this is a problem belonging to the *Method of Increments*.

2. In order to resolve a problem of this kind; let all the quantities be denoted by proper symbols, making a proper distinction between those that are constant and those that are variable; taking care to assume one variable quantity that increases uniformly; and then the other variable quantities and their various relations among one another, are to be expressed by certain equations drawn from the nature of the problem; and at last all of them expressed in terms of the quantity that increases uniformly. For which purpose

3. When there are not equations enough, it will be necessary to take the increments of some of the equations, by Prop. XI; and by that means the quantities may be compared together, and other equations derived therefrom.

4. Sometimes we must take some preceeding or successive value of an equation, according to Prop. X. from whence some new equation among the variable quantities may be deduced.

5. When

5. When quantities are not in a proper form; they may be transformed into others that are so, by Prop. XII. and then the required operations may be duly performed.

6. Sometimes a single number happens to appear in a differential equation, which must be put out, and the uniform increment put in its stead.

7. Such quantities as are superfluous and useless are to be exterminated, till you come to an equation between the increment of the quantity sought, and some quantity made up of the uniformly increasing quantity, or its increments.

8. Lastly, the integral must be found by Prop. XIII. and corrected by Prop. XIV. and your work is done.

But the practice of these rules will be better known and explained by the following problems, than can be done by many words.

P R O B L E M I.

To find the unciæ or coefficients for the binomial theorem.

Suppose $\overline{a+x}^m = a^m + r a^{m-1} x + s a^{m-2} x^2 + t a^{m-3} x^3 + v a^{m-4} x^4 \&c.$ multiply by $a+x$, and we have

$$\begin{aligned} \overline{a+x}^{m+1} &= a^{m+1} + r a^m x + s a^{m-1} x^2 + t a^{m-2} x^3 + v a^{m-3} x^4 \&c. \\ &+ 1 a^m x + r a^{m-1} x^2 + s a^{m-2} x^3 + t a^{m-3} x^4 \&c. \end{aligned}$$

And by the notation the next superior power is also,

$$\overline{a+x}^m = a^m + r a^{m-1} x + s a^{m-2} x^2 + t a^{m-3} x^3 + v a^{m-4} x^4 \&c.$$

Then comparing the homologous terms, $\overline{m} = \overline{m+1}$, and $\overline{m-m}$ or $\overline{m=1}$.

Therefore $\overline{m-1} = \overline{m}$, $\overline{m-2} = \overline{m-1}$, $\overline{m-3} = \overline{m-2}$ &c. Also $\overline{r=r+1}$, $\overline{s=s+r}$, $\overline{t=t+s}$, $\overline{v=v+t}$, &c. that is, $\overline{r-r}$ or $\overline{r=1=m}$, and $\overline{r=m}$,

and $\overline{s-s}$ or $\overline{s=r=m}$, and $\overline{s = m \frac{-1}{2}}$. Likewise $\overline{t-t}$, or $\overline{t=s = m \frac{m}{2}}$

$$\frac{m}{2}, \text{ and } t = \frac{m}{2.3}. \text{ Also } v = t = \frac{m}{2.3}, \text{ and}$$

$$v = \frac{m}{2.3.4}, \text{ \&c. Hence therefore } \overline{a+x}^m = a^m + ma^{m-1}x +$$

$$\frac{m.m-1}{1.2} a^{m-2}x^2 + \frac{m.m-1.m-2}{1.2.3} a^{m-3}x^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4} a^{m-4}x^4 \text{ \&c.}$$

And therefore if m be an affirmative integer the series will terminate.

COROLLARY.

$$\text{Hence } \overline{a+x}^m = a^m + \frac{m}{a}x A + \frac{m-1}{2} \times \frac{x}{a} B + \frac{m-2}{3} \times \frac{x}{a} C$$

$$+ \frac{m-3}{4} \times \frac{x}{a} D + \frac{m-4}{5} \times \frac{x}{a} E \text{ \&c. where } A, B, C, \text{ \&c. are}$$

the foregoing terms with their signs.

SCHOLIUM.

Although m is here $=1$, yet m may be any number whatever, whether affirmative, negative, whole or fractional; and therefore the theorem is universal.

Otherwise,

$$\text{Suppose } \overline{a+x}^m = a^m + \frac{r a^m x}{a+x} + \frac{s a^m x^2}{(a+x)^2} + \frac{t a^m x^3}{(a+x)^3} +$$

$$\frac{v a^m x^4}{(a+x)^4} \text{ \&c. to find the coefficients } r, s, t, \text{ \&c. expressed by } m. \text{ Mul-}$$

tiply both sides by $a+x$, and we have

$$\overline{a+x}^{m+1} = a^{m+1} + \frac{r a^m x}{a+x} + \frac{s a^m x^2}{(a+x)^2} + \frac{t a^m x^3}{(a+x)^3} \text{ \&c.}$$

$$+ a^m x + \frac{r a^m x^2}{a+x} + \frac{s a^m x^3}{(a+x)^2} \text{ \&c.}$$

But

But because the terms here are not homologous; multiply $a^m x + \frac{r a^m x x}{a+x}$, &c. by $a+x$, and then divide each term by $a+x$; and such terms as are not still homologous with the rest, must be again multiplied by $a+x$, and then divided by $a+x$, and so on; till at length the series will be reduced to the following form.

$$\begin{aligned} \frac{a^{m+1}}{a+x} &= a^{m+1} + \frac{r a^{m+1} x}{a+x} + \frac{s a^{m+1} x x}{a+x^2} + \frac{t a^{m+1} x^3}{a+x^3} && \&c. \\ &+ \frac{a^{m+1} x}{a+x} + \frac{r a^{m+1} x x}{a+x^2} + \frac{s a^{m+1} x^3}{a+x^3} && \&c. \\ &+ \frac{a^{m+1} x x}{a+x^2} + \frac{r a^{m+1} x^3}{a+x^3} && \&c. \\ &+ \frac{a^{m+1} x^3}{a+x^3} && \&c. \end{aligned}$$

And this, by the notation is

$$\frac{a^m}{a+x} = a^m + \frac{r a^m x}{a+x} + \frac{s a^m x x}{a+x^2} + \frac{t a^m x^3}{a+x^3} && \&c.$$

Then equating the respective quantities, $m=m+1$, or $m-m=m=1$, $r=r+1$, and $r-r$ or $r=1=m$, therefore $r=m$. After the same man-

ner $s=s$ or $s=r+1=m+m=m$, and the integral, $s = \frac{m m}{2}$. Likewise

$t=t$ or $t=s+r+1=s+s=s=\frac{m m}{2}$; and the integral $t = \frac{m m m}{2.3}$.

Again $v=v$ or $v=t+s+r+1=t+t=t=\frac{m m m}{2.3}$; and the integral

$v = \frac{m m m m}{2.3.4}$ &c. whence the series is

X

$a+x$

$$\begin{aligned} \frac{a^m}{a+x} &= a^m + \frac{m a^m x}{a+x} + \frac{m(m+1)}{2} \times \frac{a^m x^2}{a+x^2} + m \cdot \frac{m+1}{2} \cdot \frac{m+2}{3} \times \\ &\frac{a^m x^3}{a+x^3} + m \cdot \frac{m+1}{2} \cdot \frac{m+2}{3} \cdot \frac{m+3}{4} \times \frac{x^4}{a+x^4} \&c. = a^m + \frac{m x}{a+x} A + \\ &\frac{m+1}{2} \times \frac{x}{a+x} B + \frac{m+2}{3} \times \frac{x}{a+x} C + \frac{m+3}{4} \times \frac{x}{a+x} D \&c. \end{aligned}$$

Therefore if m be any negative integer, the series will terminate.

PROB. II.

If the infinitonomial $1+By+Cy^2 \&c.$ is to be raised to the m^{th} power; to find the coefficients of the terms.

$$\begin{aligned} \text{Suppose } \overline{1+By+Cy^2+Dy^3+Ey^4 \&c.}^m \\ &= \left. \begin{aligned} 1+rBy+sB^2 \\ +tC \end{aligned} \right\} y^2 + \left. \begin{aligned} +nB^3 \\ +wBC \\ +xD \end{aligned} \right\} y^3 + \left. \begin{aligned} +\alpha B^4 \\ +\beta B^2C \\ +\gamma CC \\ +\delta BD \\ +\varepsilon E \end{aligned} \right\} y^4 \&c. \end{aligned}$$

Multiply this by

$$1+By+Cy^2+Dy^3+Ey^4 \&c.$$

And we shall have

$$\begin{aligned} &\overline{1+By+Cy^2+Dy^3+Ey^4}^{m+1} \\ &= \left. \begin{aligned} 1+rBy+sB^2 \\ +tC \end{aligned} \right\} y^2 + \left. \begin{aligned} +uB^3 \\ +wBC \\ +xD \end{aligned} \right\} y^3 + \left. \begin{aligned} +\alpha B^4 \\ +\beta B^2C \\ +\gamma CC \\ +\delta BD \\ +\varepsilon E \end{aligned} \right\} y^4 \&c. \\ &+ By + rB^2y^2 + \left. \begin{aligned} +sB^3 \\ +tCB \end{aligned} \right\} y^3 + \left. \begin{aligned} +uB^4 \\ +wB^2C \\ +xB^2D \end{aligned} \right\} y^4 \\ &+ Cy^2 + rBCy^3 + \left. \begin{aligned} +sB^2C \\ +tCC \end{aligned} \right\} y^4 \\ &+ Dy^3 + rBDy^4 + Ey^4 \\ &= \end{aligned}$$

= by the notation to

$$\left. \begin{array}{l} 1 + r B y + s B^2 \\ + t C \end{array} \right\} y^2 \quad \left. \begin{array}{l} + u B^3 \\ + w BC \\ + x D \end{array} \right\} y^3 \quad \left. \begin{array}{l} + \alpha B^4 \\ + \beta B^2 C \\ + \gamma CC \\ + \delta BD \\ + \epsilon E \end{array} \right\} y^4 \text{ \&c.}$$

$$= 1 + B y + C y^2 + D y^3 \Big|_1^m.$$

Here $m = m + 1$, and $m = m$ or $m = 1$. And comparing the homologous terms,

$$r = r + 1, \text{ and } r = r \text{ or } r = 1 = m, \text{ and } r = m.$$

$$\text{In like manner } s = s \text{ or } s = r = m, \text{ and } s = \frac{m \ m}{2}.$$

$$\text{Also } t = t \text{ or } t = 1 = m, \text{ and } t = m.$$

$$\text{Also } u = u \text{ or } u = s = \frac{m \ m}{2}, \text{ and } u = \frac{m \ m \ m}{2 \cdot 3}.$$

$$\text{Also } w = w \text{ or } w = r + t = 2m, \text{ and } w = \frac{2 \ m \ m}{2} = m \ m.$$

$$\text{Likewise } x = x \text{ or } x = 1 = m, \text{ and } x = m.$$

$$\text{Again } \alpha = \alpha \text{ or } \alpha = u = \frac{m \ m \ m}{2 \cdot 3}, \text{ and } \alpha = \frac{m \ m \ m \ m}{2 \cdot 3 \cdot 4};$$

$$\text{And } \beta = w + s = m \ m + \frac{m \ m}{2} = \frac{3 \ m \ m}{2}, \text{ and } \beta = \frac{m \ m \ m}{2};$$

$$\text{Also } \gamma = t = m, \text{ and } \gamma = \frac{m \ m}{2}.$$

$$\text{Also } \delta = r + x = 2m, \text{ and } \delta = m \ m.$$

$$\text{And } \epsilon = 1 = m, \text{ and } \epsilon = m,$$

&c.

Whence

Whence $\overline{1 + By + Cy^2 + Dy^3 \&c.}^m$

$$\begin{aligned}
 & 1 + mBy + m \cdot \frac{m-1}{2} B^2 \left. \vphantom{\frac{m-1}{2}} \right\} y^2 + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} B^3 \left. \vphantom{\frac{m-2}{3}} \right\} y^3 \\
 & \quad + mC \left. \vphantom{\frac{m-1}{2}} \right\} + m \cdot \frac{m-1}{2} BC \left. \vphantom{\frac{m-1}{2}} \right\} \\
 & \quad + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} B^4 \left. \vphantom{\frac{m-1}{2}} \right\} y^4 \&c. \\
 & \quad + m \cdot \frac{m-1}{2} B^2 C \left. \vphantom{\frac{m-1}{2}} \right\} \\
 & \quad + m \cdot \frac{m-1}{2} CC \left. \vphantom{\frac{m-1}{2}} \right\} \\
 & \quad + m \cdot \frac{m-1}{2} BD \left. \vphantom{\frac{m-1}{2}} \right\} \\
 & \quad + mE \left. \vphantom{\frac{m-1}{2}} \right\}
 \end{aligned}$$

PROB. III.

If a series of quantities be given; to find a general series for the value of any term, expressed by the first of the first, second, third, &c. differences of these terms.

Series	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>
1 diff.		<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>
2 diff.			<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
3 diff.				<i>a</i>	<i>b</i>	<i>c</i>
4 diff.					<i>a</i>	<i>b</i>
						<i>&c.</i>

These differences are found, by subtracting each term from the following one.

Here $b = a + a$

And $c = b + b = a + a + b = a + 2a + a$

Also $d = c + c = a + 2a + a + c$, and for the same reason that $c = a + 2a + a$, will $c = a + 2a + a$, and therefore $d = a + 3a + 3a + a$, and so of others.

Therefore

Therefore let the n^{th} term be expressed in general by the series $a + Aa + Ba + Ca + Da \&c.$ then the $\overline{n+1}^{\text{th}}$ term will be

$$\left. \begin{array}{l} a + Aa + Ba + Ca + Da \\ + a + Aa + Ba + Ca \end{array} \right\} \&c. \text{ And this by the notation is}$$

equal to, $a + Aa + Ba + Ca + Da \&c.$

Here $a=a$, $A=A+1$, $B=B+A$, $C=C+B$, $\&c.$ that is, $A=A$ or $A=1=n$, and $A=n$, which needs no correction; also $B=A=n$, and

$$B = \frac{n}{2}. \text{ And } C = B = \frac{n}{2}, \text{ and } C = \frac{n}{2.3}. \text{ And } D = C = \frac{n}{2.3}, \text{ whence } D = \frac{n}{2.3.4} \&c.$$

Therefore the n^{th} term of the series is,

$$a + na + \frac{n \cdot \overline{n-1}}{2} a + \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2.3} a + \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}}{2.3.4} a \&c. = a + na + \frac{n-1}{2} P a + \frac{n-2}{3} Q a + \frac{n-3}{4} R a \&c. \text{ where } P, Q, R, S, \text{ are the foregoing coefficients; and } a, a, a \&c. \text{ the first of the several differences.}$$

P R O B. IV.

To find any horizontal line or row of Pascalius's arithmetical triangle:

I	I	I	I	I	I	&c.
	I	2	3	4	5	
		I	3	6	10	
			I	4	10	
				I	5	
					I	

The construction of this triangle is thus: To any number in the given row, add that above it, the sum is the next number in the same row.

Y

Therefore

Therefore put $1, a, b, c, d, \&c.$ for the numbers in the n^{th} row, and those in the $\overline{n+1}^{\text{th}}$ row will be $1, a, b, c, d, \&c.$ Thus in general,

1	1	1	1	1	1	1	1	&c.
	1	2	3	4	5	6	7	
		1	3	6	10	15	21	
			1	a	b	c	d	
				1	a	b	c	
					1	1	1	
						&c.		

Here $n=1$, and by the construction, $1+a=a$; $a+b=b$; $b+c=c$; $c+d=d$; &c. whence $a-a=1=n$, or $a=n$, and $a=n$. Also $b-b$ or $b=a=n$; and $b = \frac{n \cdot n}{2}$. Also $c=b=\frac{n \cdot n}{2}$, and $c = \frac{n \cdot n \cdot n}{2 \cdot 3}$. And

$d=c=\frac{n \cdot n \cdot n}{2 \cdot 3}$; and $d = \frac{n \cdot n \cdot n \cdot n}{2 \cdot 3 \cdot 4}$, &c.

Whence the n^{th} row is,

$$= 1 + n + \frac{n \cdot \overline{n+1}}{2} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} + n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} \&c.$$

$$= 1 + n + \frac{n+1}{2} A + \frac{n+2}{3} B + \frac{n+3}{4} C + \frac{n+4}{5} D, \&c.$$

PROB. V.

To find a general series expressing the figurate numbers of any order.

Figurate or polygonal numbers are several ranks of numbers, beginning from a series of units. And any term or number in any rank is found, by adding together all the numbers in the preceeding rank, to that place; or by adding that on the left-hand to that above it.

	1	1	1	1	1	1	&c.
	1	2	3	4	5	6	
	1	3	6	10	15	21	
	1	4	10	20	35	56	
In general	1	a	b	c	d	e	
	1	a	b	c	d	e	
		1	1	1	1	1	
			&c.				

Let

Let the series of numbers in the n^{th} rank be denoted by $1, a, b, c, \&c.$; and in the $\overline{n+1}^{\text{th}}$ rank by $1, a, b, c, \&c.$ to find $a, b, c, \&c.$

By the construction of the table, $1+a=a$, and $a-a$ or $a=1=n$, and $a=n$, the integral.

Also $1+a+b=b$, or $b=1+a=a=n$, and $b=\frac{n n}{2}$, the integral.

Again $1+a+b+c=c$, and $c=1+a+b=b=\frac{1 \cdot 2}{2}$, and the integral is

$$c = \frac{n n n}{2 \cdot 3}.$$

Likewise $1+a+b+c+d=d$, and $d=1+a+b+c=c=\frac{1 \cdot 2 \cdot 3}{2 \cdot 3}$, and

the integral is $d = \frac{n n n n}{2 \cdot 3 \cdot 4}.$

Also $e=1+a+b+c+d=d=\frac{1 \cdot 2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 4}$, and $e = \frac{n n n n n}{2 \cdot 3 \cdot 4 \cdot 5} \&c.$ but

$n n = n \cdot \overline{n+1}$; and $n n n = n \cdot \overline{n+1} \cdot \overline{n+2}$; and $n n n n =$

$n \cdot \overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3}$, &c. therefore the numbers of any order (n), are severally expressed by this series,

$$1, n, n \cdot \frac{n+1}{2}, n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}, n \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4}, \&c.$$

Or by this,

$$1, n, \frac{n+1}{2} A, \frac{n+2}{3} B, \frac{n+3}{4} C, \frac{n+4}{5} D, \frac{n+5}{6} E, \&c.$$

PROB. VI.

Let the following rows or series of quantities be given; whose numeral coefficients are found by this rule, to the n^{th} term of any row, add the $n-1^{\text{th}}$ term of the foregoing row, the sum is the n^{th} term of the next following row.

$$\begin{aligned} &1 \\ &m \\ &m \quad m-1 \\ &m^2-2m \\ &m^3-3m \quad m+1 \\ &m^4-4m^2+3m \\ &m^5-5m^3+6m^2-1, \text{ \&c. and} \end{aligned}$$

Let $m^n - am^{n-2} + b m^{n-4} - c m^{n-6} + d m^{n-8} \text{ \&c.}$ express any row in general. Then by the construction, the next row is

$$\begin{aligned} m^{n+1} &= \overline{a+1} \cdot m^{n-1} + \overline{b+a} \cdot m^{n-3} - \overline{c+b} \cdot m^{n-5} \text{ \&c. is} \\ &= m^n - a m^{n-2} + b m^{n-4} - c m^{n-6} \text{ \&c.} \end{aligned}$$

Then comparing the coefficients, &c. $n=n+1$, and $n=n$ or $n=1$. Also $a=a+1$, and $a=a$ or $a=1=n$, and $a=n$. For when $n=2$,

$a=1=n-1=n$. Again, $b=b$ or $b=a=n$; and $b=\frac{n \cdot n}{2}$; for when

$n=4$, $b=1$. Again, $c=c$ or $c=b=\frac{n \cdot n}{2}$, and $c=\frac{n \cdot n \cdot n}{2 \cdot 3}$. In

like manner $d=c=\frac{n \cdot n \cdot n}{2 \cdot 3}$, and $d=\frac{n \cdot n \cdot n \cdot n}{2 \cdot 3 \cdot 4}$, &c. consequently the series required is,

$$\begin{aligned} &m^n - \overline{n-1} \cdot m^{n-2} + \frac{\overline{n-2} \cdot \overline{n-3}}{2} m^{n-4} - \frac{\overline{n-3} \cdot \overline{n-4} \cdot \overline{n-5}}{1 \cdot 2 \cdot 3} m^{n-6} \\ &+ \frac{\overline{n-4} \cdot \overline{n-5} \cdot \overline{n-6} \cdot \overline{n-7}}{1 \cdot 2 \cdot 3 \cdot 4} m^{n-7} \text{ \&c.} \end{aligned}$$

PROB.

PROB. VII.

Having given the sine of an arch; to find the sine of n times the arch.

Let radius $=1$, the sine of the single arch $=s$, the cosine $=c$; and the cosine of twice the arch will be $2cs$. And by trigonometry, as radius : to twice the cosine of an arch :: so the sine of n times the arch : to the sum of the sines of $n-1$ and $n+1$ times that arch, from hence we get the following series,

1	s
2	$c \times 2s$
3	$3s - 4s^3$
4	$c \times 4s - 8s^3$
5	$5s - 20s^3 + 16s^5$
6	$c \times 6s - 32s^3 + 32s^5$
7	$7s - 56s^3 + 112s^5 - 64s^7$
8	$c \times 8s - 80s^3 + 192s^5 - 128s^7$
	&c.

It is evident, all the even sines are affected with c , and all the odd ones (arising from the multiplication of $1-s$ instead of cc), consist only of the powers of s . Hence there are two cases for the odd and the even numbers.

CASE I.

To find the sine of 1, 3, 5, 7, &c. times the arch A . By the principles of trigonometry, if 3 arches are in arithmetic progression, twice the sine of the mean arch $\times$ cosine of the common difference, is $=$ sum of the sines of the extremes. Here the common difference is $2A$, whose cosine is $1-2ss$. Therefore let the sine of the n^{th} arch be

$$\begin{array}{l}
 \text{is} \quad ns - as^3 + bs^5 - cs^7 + ds^9 \text{ \&c. twice this sine} \\
 \text{mult. by} \quad 1-2ss \\
 \hline
 \begin{array}{l}
 2ns - 2as^3 + 2bs^5 - 2cs^7 + 2ds^9 \text{ \&c.} \\
 - 4ns^3 + 4as^5 - 4bs^7 + 4cs^9 \\
 \hline
 2ns - 2as^3 + 2bs^5 - 2cs^7 + 2ds^9 \text{ \&c.} \\
 - 4ns^3 + 4as^5 - 4bs^7 + 4cs^9 \\
 \hline
 \end{array} = \text{sum of the ex-} \\
 \text{trems} = \left\{ \begin{array}{l} ns - as^3 + bs^5 - cs^7 + ds^9 \\ + ns - as^3 + bs^5 - cs^7 + ds^9 \end{array} \right\} \text{ \&c.}
 \end{array}$$

Z

Here

Here $n=2$, and equating the coefficients, $2n = n + n = 2n$, $2a + 4n = a + a$; and taking the next succeeding values $2a + 4n = a + a$, that is, $2a + 2a + 4n$

$= a + 2a + a + a$, and $a = 4n$; and the integral is $a = \frac{4nn}{2n} + 1 = nn + 1$,

for when $n=1$, $a=4$. And the integral taken again is $a = \frac{nnn}{3n} +$

$\frac{n}{n}$ for when $n=3$, $a=4$.

Again $2b + 4a = b + b$, and $2b + 4a = b + b$, that is $2b + 2b + 4a = b +$

$2b + b + b$, or $b = 4a = \frac{4nnn}{3n} + \frac{4n}{n}$, and the integral $b = \frac{4nnnn}{4 \cdot 3nn} +$

$\frac{4nn}{2nn} - \frac{1}{4}$, for when $n=3$, $b=16$, and $b = \frac{4n \dots n}{4 \cdot 3 \cdot 5n^2} + \frac{4nnn}{2 \cdot 3n^2} -$

$\frac{n}{4n}$, for when $n=5$, $b=16$. &c.

Or thus, to avoid these compound quantities, since we have $a = nn + 1 = n \times n + n + 1 = nn + 2n + 1 = n + 1^2$, put $m = n + 1$, then $m = n = 2$, and $a = mm = m + m$, and the integral

$a = \frac{mmm}{3m} + \frac{mm}{2}$, which is correct; for when $n=5$, $m=6$, and $a=20$.

Again, $b = 4a = \frac{4mmmm}{2 \cdot 3} + 2mm$, and the integral $b = \frac{4mm \dots m}{2 \cdot 3 \cdot 4m} + \frac{2mmmm}{3m}$, correct; and $b = \frac{4mm \dots m}{2 \cdot 3 \cdot 4 \cdot 5mm} + \frac{2mm \dots m}{2 \cdot 3 \cdot 4m} =$

$\frac{m \dots m}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{m \dots m}{2 \cdot 3 \cdot 4}$, correct.

Likewise

Likewise $c = 4b = \frac{4 \frac{m \dots m}{-2}}{2.3.4.5} + \frac{4 \frac{m \dots m}{-1}}{2.3.4}$, and $c = \frac{4 \frac{m \dots m}{-3}}{2..6m} + \frac{4 \frac{m \dots m}{-2}}{2..5m}$, and $c = \frac{m \dots m}{-4} + \frac{m \dots m}{-3}$, correct.

And $d = 4c$, and $d = \frac{4 \frac{m \dots m}{-4}}{2..8m} + \frac{4 \frac{m \dots m}{-3}}{2..7m}$, and $d = \frac{m \dots m}{-5} + \frac{m \dots m}{-4}$; correct: &c. Whence

$$a = \frac{m}{-2} + 3 \times \frac{m}{2.3}, b = \frac{m}{-3} + 5 \times \frac{m}{2.3.4.5}, c = \frac{m}{-4} + 7 \times \frac{m}{2.3.4.5.6.7},$$

$$d = \frac{m}{-5} + 9 \times \frac{m}{2.3.4.5.6.7.8.9}; \text{ \&c. but } n = \frac{m}{-2} + 3 = \frac{m}{-3} + 5 = \frac{m}{-4} + 7 = \frac{m}{-5} + 9 \text{ \&c. and } m, m, m \text{ \&c. } = n-1, n-3, n-5 \text{ \&c. Also } m, m, m, \text{ \&c. } = n+1, n+3, n+5 \text{ \&c. Whence}$$

$$a = n \cdot \frac{n-1}{2} \cdot \frac{n+1}{3}, b = n \cdot \frac{n-3}{2} \cdot \frac{n-1}{3} \cdot \frac{n+1}{4} \cdot \frac{n+3}{5}, c = n \cdot \frac{n-5}{2} \cdot \frac{n-3}{3} \cdot \frac{n-1}{4} \cdot \frac{n+1}{5} \cdot \frac{n+3}{6} \cdot \frac{n+5}{7}, \text{ \&c. Hence when } n \text{ is an odd number the sine of } nA =$$

$$ns = \frac{n \cdot nn-1}{2.3} s^3 + \frac{n \cdot nn-1 \cdot nn-9}{2..5} s^5 + \frac{n \cdot nn-1 \cdot nn-9 \cdot nn-15}{2...7} s^7 + \frac{n \cdot nn-1 \cdot nn-9 \cdot nn-25 \cdot nn-49}{2...9} s^9 \text{ \&c.}$$

CASE II.

To find the sine of 2, 4, 6, 8, &c. times the arch A.

Let the sine of the n^{th} arch be

$$c \times ns - as^3 + bs^5 + cs^7 + ds^9 \text{ \&c.}$$

Then preceeding in all respects as before till we get $a = 4n$, take the
integral

integral which is $a = \frac{4nn}{2n} = nn$, which is correct, for when $n=2$,

$a=8$. And taking the integral again $a = \frac{nnn}{3n} = \frac{nn}{2.3}$, correct, for when $n=4$, $a=8$.

Again, we shall find as before $b = 4a = \frac{4nnn}{2.3}$, and the integral

$$b = \frac{4nnnn}{2.3.4n}, \text{ and } b = \frac{4n \dots n}{2.3.4.5nn} = \frac{n \dots n}{2.3.4.5}, \text{ correct.}$$

$$\text{Again, } c = 4b = \frac{4n \dots n}{2 \dots 5}, \text{ and } c = \frac{4n \dots n}{2 \dots 6n}, \text{ and } c = \frac{4n \dots n}{2 \dots 7nn} \\ = \frac{n \dots n}{2 \dots 7}, \text{ correct.}$$

$$\text{Also } d = 4c = \frac{4n \dots n}{2 \dots 7}, \text{ and } d = \frac{4n \dots n}{2 \dots 8n}, \text{ and } d = \frac{4n \dots n}{2 \dots 9nn} \\ = \frac{n \dots n}{2 \dots 9}, \text{ \&c. Therefore when } n \text{ is an even number, the sine of}$$

$$nA = \\ c \times ns = \frac{n \cdot nn - 4}{2.3} s^3 + \frac{n \cdot nn - 4 \cdot nn - 16}{2.3.4.5} s^5 - \frac{n \cdot nn - 4 \cdot nn - 16 \cdot nn - 36}{2 \dots 7} s^7 \\ + \frac{n \cdot nn - 4 \cdot nn - 16 \cdot nn + 36 \cdot nn - 64}{2 \dots 9} s^9 \text{ \&c.}$$

PROB. VIII.

The cofine of an arch A being given; to find the cofine of n times that arch.

Let $x = \text{cofine}$, $\text{rad} = 1$. Then by trigonometry, as $\text{rad} : \text{twice the cofine} :: \text{cof. } nA : \text{cof. } n-1.A + \text{cof. } n+1.A$.

Hence we have the following series of cofines.

$$\begin{array}{rcl}
 0 & 1 & \\
 1 & x & \\
 2 & -1 + 2xx & \\
 3 & -3x + 4x^3 & \\
 4 & 1 - 8xx + 8x^4 & \\
 5 & 5x - 20x^3 + 16x^5 & \\
 6 & -1 + 18xx - 48x^4 + 32x^6 & \\
 7 & -7x + 56x^3 - 112x^5 + 64x^7 & \\
 8 & 1 - 32x^2 + 160x^4 - 256x^6 + 128x^8 & \\
 & \&c. &
 \end{array}$$

It is plain, all the even cosines begin with 1, and all the odd ones with n , and hence arise two cases of this problem, for the odd and even numbers. And both cases will be solved by help of this theorem of trigonometry: If 3 arches be in arithmetic progression, the cos. mean arch $\times$ twice the cos. common difference = sum of the cosines of the extremes. And the cos. common difference = $2 \times n - 1$. Therefore in

CASE I. For the odd numbers.

Let cof. $nA = nx - ax^3 + bx^5 - cx^7 + dx^9 \quad \&c.$
 mult. by $-2+4xx$

$$\begin{array}{r} \hline -2nx + 2ax^3 - 2bx^5 + 2cx^7 - 2dx^9 \} \&c. \\ + 4nx^3 - 4ax^5 + 4bx^7 - 4cx^9 \} \\ \hline = \\ -\underset{\underset{1}{|}}{n}x + \underset{\underset{1}{|}}{a}x^3 - \underset{\underset{1}{|}}{b}x^5 + \underset{\underset{1}{|}}{c}x^7 - \underset{\underset{1}{|}}{d}x^9 \} \\ \text{Sum of ex-} \} \\ \text{trems} \} \end{array} \&c.$$

Now equating the coefficients, $2n = n + n = 2n$. Also $2a + 4n = a + a$, or taking the next succeeding values, $2a + 4n = a + a$, that is, $2a + 2a + 4n = 2a + 2a + a$, and $a = 4n$, and $a = \frac{4nn}{2n} = n$, because $n = 2$, and corrected $a = n + 1$, for when $n = 1$, $a = 4$. Put $m = n + 1$, then

$$a = mm = mm + mm, \text{ and } a = \frac{\overset{m}{-2} \overset{m}{-1} \overset{m}{-}}{3m} + \frac{\overset{m}{-1} \overset{m}{-}}{2}, \text{ correct.}$$

A a

Again,

Again, $2b + 4a = b + b$, whence $b = 4a$; likewise $c = 4b$; &c. All which equations being the same as in Case I. of the last Problem, give the same values of a, b, c , &c. and consequently the series will be the same, that is, when n is an odd number; the cof. nA will be + or - this series,

$$n x - \frac{n \cdot nn - 1}{2 \cdot 3} x^3 + \frac{n \cdot nn - 1 \cdot nn - 9}{2 \cdot \cdot 5} x^5 - \frac{n \cdot nn - 1 \cdot nn - 9 \cdot nn - 25}{2 \cdot \cdot \cdot 7} x^7 + \&c.$$

CASE II. When n is an even number.

Let cof. nA be $= 1 - ax^2 + bx^4 - cx^6 + dx^8 \&c.$

Proceeding the same way as in Case I, you will have the same equations for a, b, c, d , &c. as before, and first $a = 4$, and $a = \frac{4n}{n} = 2n$, and corrected $a = 2n + 2$, for when $n = 2$, $a = 6$,

$$\text{And } a = \frac{2nn}{2n} + \frac{2n}{n} = \frac{nn}{2} + n, \text{ correct.}$$

$$\text{Then } b = 4a = \frac{4nn}{2} + 4n, \text{ and } b = \frac{4nnn}{2 \cdot 3n} + \frac{4nn}{2n} = \frac{nnn}{3} + nn;$$

$$\text{and } b = \frac{nnnn}{3 \cdot 4n} + \frac{nnn}{3n} = \frac{nnnn}{2 \cdot 3 \cdot 4} + \frac{nnn}{2 \cdot 3}, \text{ correct.}$$

$$\text{Again } c = 4b = \frac{4nn \cdot n}{2 \cdot 3 \cdot 4} + \frac{4nn \cdot n}{2 \cdot 3}, \text{ then } c = \frac{4nn \cdot n}{2 \cdot \cdot 5n} + \frac{4nn \cdot n}{2 \cdot \cdot 4n};$$

$$\text{and } c = \frac{4nn \cdot n}{2 \cdot \cdot 6nn} + \frac{4nn \cdot n}{2 \cdot \cdot 5nn} = \frac{nn \cdot n}{2 \cdot \cdot 6} + \frac{nn \cdot n}{2 \cdot \cdot 5}, \text{ correct.}$$

$$\text{Also } d = 4c = \frac{4nn \cdot n}{2 \cdot \cdot 6} + \frac{4nn \cdot n}{2 \cdot \cdot 5}, \text{ and } d = \frac{4nn \cdot n}{2 \cdot \cdot 7n} + \frac{4nn \cdot n}{2 \cdot \cdot 5n}; \text{ and}$$

$$d = \frac{4nn \cdot n}{2 \cdot \cdot 8nn} + \frac{4nn \cdot n}{2 \cdot \cdot 6nn} = \frac{nn \cdot n}{2 \cdot \cdot 8} + \frac{nn \cdot n}{2 \cdot \cdot 7}, \text{ correct. } \&c.$$

But

$$\text{But } a = \frac{n+2}{2} n = \frac{n-2+2}{2} n = \frac{n n}{2}.$$

$$b = \frac{n-4+4}{2 \cdot 3 \cdot 4} \times \overline{n-2} \cdot n \cdot \overline{n+2} = \frac{n n \cdot \overline{nn-4}}{2 \cdot 3 \cdot 4}$$

$$c = \frac{n-6+6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \overline{n-4} \cdot \overline{n-2} \cdot n \cdot \overline{n+2} \cdot \overline{n+4} =$$

$$\frac{n n \cdot \overline{nn-4} \cdot \overline{nn-16}}{2 \dots 6}, \text{ and } d = \frac{n n \cdot \overline{nn-4} \cdot \overline{nn-16} \cdot \overline{nn-36}}{2 \dots 8}, \text{ \&c.}$$

And therefore when n is an even number, the cosine of the n^{th} arch will be $= +$ or $-$ this series,

$$1 - \frac{n n}{2} x^2 + \frac{n n \cdot \overline{nn-4}}{2 \cdot 3 \cdot 4} x^4 - \frac{n n \cdot \overline{nn-4} \cdot \overline{nn-16}}{2 \dots 6} x^6 +$$

$$\frac{n n \cdot \overline{nn-4} \cdot \overline{nn-16} \cdot \overline{nn-36}}{2 \dots 8} - \text{\&c.}$$

P R O B. IX.

The versed sine of an arch being given, to find the versed sine of n times the arch.

Let $\text{rad.} = 1$, $v =$ versed sine of the arch A . By trigonometry; if 3 arches are in arithmetic progression the $\text{cos. mean arch} \times \text{cos. common difference} =$ half the sum of the cosines of the extream arches; and $1-v$ is the cosine of the common difference; for cosine $= 1-v$.

$$\text{Therefore } 1-v \times 1 - \text{vers. } nA = \frac{1 - \text{vers. } n-1.A + 1 - \text{vers. } n+1.A}{2},$$

which reduced is $2v + 2 - 2v \times \text{vers. } nA = \text{vers. } n-1.A + \text{vers. } n+1.A$. Hence we have the following series of versed sines,

1	v				
2	$4v - 2vv$				
3	$9v - 12vv + 4v^3$				
4	$16v - 40vv + 32v^3 - 8v^4$				
5	$25v - 100vv + 140v^3 - 80v^4 + 16v^5$				
6	$36v - 210vv + 448v^3 - 432v^4 + 192v^5 - 32v^6$				
	$\text{\&c.}$				

In

But $a = \frac{n}{1} + n = \frac{n}{1}$.

$$b = \frac{n+2}{2.3} \times \frac{n}{1} = \frac{n \cdot n-1}{2.3}$$

$$c = \frac{n+3}{3.5.6} \times \frac{n}{2} = \frac{n \cdot n-1 \cdot n-4}{3.5.6}$$

$$d = \frac{n \cdot n-1 \cdot n-4 \cdot n-6}{3.4.5.6.7}$$

&c.

Whence the verf. $nA =$

$$\frac{n \cdot n-1}{2.3} P v - \frac{n-4}{3.5} Q v - \frac{n-9}{4.7} R v - \frac{n-16}{5.9} S v - \frac{n-25}{6.11} T v, \text{ \&c. } P, Q, R, \text{ \&c. being the foregoing terms.}$$

PROB. X.

The cord of an arch A being given; to find the cord of $n \times A$.

Let rad. = 1, cord of $A = x$, and $C =$ cord of the supplement. Then by trigonometry, $1 : C :: \text{cord } nA : \text{cord } n-1.A + \text{cor. } n+1.A$. Therefore any cord $\times$ by C , and the foregoing subtracted, gives the next following cord; observing, instead of CC to multiply by $-xx+4$, its equal, and from hence is had the following series of cords.

1	x
2	$C \times x$
3	$-x^3 + 3x$
4	$C \times -x^3 + 2x$
5	$x^5 - 5x^3 + 5x$
6	$C \times x^5 - 4x^3 + 3x$
7	$-x^7 + 7x^5 - 14x^3 + 7x$
8	$C \times -x^7 + 6x^5 - 10x^3 + 4x$
	&c.

Hence all the even cords only are affected with C , which makes two cases for the odd and even cords. And by trigonometry; if 3 arches are in arithmetic progression, the cord of the mean arch $\times$ cord of the sup. of the common difference = radius $\times$ half the sum of the cords of the extrem arches. And in the even, as well as in the odd arches, the

B b

common

common difference is $2A$; and the cord of its supplement;
 $\sqrt{4-CCxx} = 2 - xx$.

CASE I. For odd arches.

For the cord of the n^{th} arch, assume

$$x^n - ax^{n-2} + bx^{n-4} - cx^{n-6} \&c.$$

mult. by $-xx + 2$

$$\left. \begin{array}{l} -x^{n+2} + ax^n - bx^{n-2} + cx^{n-4} \\ + 2x^n - 2ax^{n-2} + 2bx^{n-4} \end{array} \right\}$$

$$= \left. \begin{array}{l} -x^{n+2} + ax^n - bx^{n-2} + cx^{n-4} \\ -x^{n-2} + ax^{n-4} \end{array} \right\}$$

&c. = sum of the extremes.

And comparing the coefficients, $a=a+2$, and $a-a$ or $a=2=n$, and $a=n$, correct.

$$\text{Again, } b+1=b+2a, \text{ and } b=2a-1=2n-1, \text{ and } b = \frac{2nn}{2n} - \frac{n}{n}$$

$$= \frac{nn-n}{2}, \text{ correct.}$$

$$\text{Again, } c+a=c+2b, \text{ and } (c-c)=c=2b-a=nn-n-n=nn-2n+2;$$

$$\text{and } c = \frac{nnn}{3n} - \frac{2nn}{2n} + \frac{2n}{n} = \frac{nnn}{2.3} - \frac{n}{2} + n.$$

$$\text{In like manner, } d+2c=d+b, \text{ or } d=2c-b, \text{ and } e=2d-c; \text{ and}$$

$$f=2e-d; \&c. \text{ whence } d = \frac{nnn}{3} - \frac{3nn}{2} + \frac{9n}{2} - 5, \text{ and}$$

$$d = \frac{n \dots n}{3.4n} - \frac{3nnn}{2.3.n} + \frac{9nn}{2.2n} - \frac{5n}{n}. \text{ Also } e = \frac{n \dots n}{3.4} -$$

$$\frac{2nnnn}{3} + \frac{15nn}{4} - 12n + 14; \text{ and } e = \frac{n \dots n}{2.3.4.5} - \frac{n \dots n}{3.4} + \frac{15nnn}{2.3.4}$$

$$- 3nn + 7n. \&c.$$

But

But $a=n$

$$b = \frac{n \cdot n-1}{2} = n \cdot \frac{n-3}{2}$$

$$c = n \cdot \frac{n-4 \cdot n-5}{2 \cdot 3}$$

$$d = n \cdot \frac{n-5 \cdot n-6 \cdot n-7}{2 \cdot 3 \cdot 4}$$

$$e = n \cdot \frac{n-6 \cdot n-7 \cdot n-8 \cdot n-9}{2 \cdot 3 \cdot 4 \cdot 5}$$

&c.

And the cord of nA will = + or - this series,

$$x^n - n x^{n-2} + \frac{n \cdot n-3}{2} x^{n-4} - \frac{n \cdot n-4 \cdot n-5}{2 \cdot 3} x^{n-6} \text{ &c. when } n \text{ is odd.}$$

CASE II. When n is even.

Let the cord of the n^{th} arch be

$C \times x^{n-1} - a x^{n-3} + b x^{n-5} - c x^{n-7} \text{ &c.}$ This multiplied by $-xx+2$, gives the same general equation as before; whence comparing the coefficients, we have the same equations for a, b, c , &c. as in Case I, viz. $a=n$, $b=2a-1$, $c=2b-a$, $d=2c-b$, &c. And finding the integrals, $a=n$, for when $n=4$, a or $n=2$.

$$\text{Again, } b=2n-1, \text{ and } b = \frac{2 \cdot n \cdot n}{2 \cdot n} - \frac{n}{n} + 2 = \frac{n \cdot n}{2} + \frac{3 \cdot n}{2}.$$

$$\text{Also } c = \frac{n \cdot n}{2 \cdot 3} + \frac{2 \cdot n}{2}, \text{ and } c = \frac{n \cdot n \cdot n}{2 \cdot 3} + \frac{n \cdot n}{2}.$$

$$\text{And } d = \frac{n \cdot n \cdot n}{3} + \frac{5 \cdot n \cdot n}{2} + \frac{5 \cdot n}{2}, \text{ and}$$

$$d = \frac{n \cdot n \cdot n}{2 \cdot 3 \cdot 4} + \frac{5 \cdot n \cdot n \cdot n}{3 \cdot 4} + \frac{5 \cdot n \cdot n}{2 \cdot 4} \text{ &c.}$$

But

But $a = n = n-2$.

$$b = \frac{n}{2} \times \frac{n}{2} + \frac{3}{2} \times \frac{n}{2} = \frac{n-6+3}{2} \times \frac{n-4}{2} = \frac{n-4}{2} \times \frac{n-3}{2}.$$

$$c = \frac{n+3}{2 \cdot 3} \times \frac{n}{2} \times \frac{n}{2} = \frac{n-4}{2 \cdot 3} \times \frac{n-5}{2} \times \frac{n-6}{2}.$$

$$d = \frac{n-5}{2 \cdot 3 \cdot 4} \times \frac{n-6}{2} \times \frac{n-7}{2} \times \frac{n-8}{2} \dots \&c.$$

Hence, when n is an even number, the cord of the arch nA , will be + or - this series,

$$C \times x^{n-1} - \frac{n-2}{2} x^{n-3} + \frac{n-3}{2} \times \frac{n-4}{2} x^{n-5} - \frac{n-4}{2 \cdot 3} \times \frac{n-5}{2} \times \frac{n-6}{2} x^{n-7} \\ + \frac{n-5}{2 \cdot 3 \cdot 4} \times \frac{n-6}{2} \times \frac{n-7}{2} \times \frac{n-8}{2} x^{n-9} \dots \&c.$$

PROB. XI.

To find the general construction of any row in Table I, Prop. XII.

Put $a, b, c, d, \&c.$ for the numbers in the n^{th} row, then

Row	1	1	1	1	1	1	1	1	&c.
2									
n			3	7	15	31	63		
$n+1=n$			a	b	c	d	e		
				a	b	c	d		

By the construction $0+a=a$, $na+b=b$, $nb+c=c$, $nc+d=d$, &c. Therefore $a=0$, $b=na$, $c=nb$, $d=nc$, $e=nd$, &c.

$$\text{Therefore } a=1, b=n, \text{ and } b = \frac{n}{2} \times \frac{n}{2} \dots c = nb = \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} =$$

$$\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}, c = \frac{n}{2 \cdot 4} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \dots d = nc = \frac{n}{2 \cdot 3} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} =$$

$$\frac{n}{3} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}, \text{ and } d = \frac{n}{2 \cdot 4 \cdot 6} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} + \frac{n}{4} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}.$$

In like manner,

$$e = \frac{n \dots n}{8.2.4.6} - \frac{n \dots n}{4.6} + \frac{13 n \dots n}{6.3.4} - \frac{n \dots n}{5}$$

$$f = \frac{n \dots n}{2.4.6.8.10} - \frac{n \dots n}{2.3.4.6} + \frac{17 n \dots n}{2.3.6.8} - \frac{11 n \dots n}{3.4.5} + \frac{n \dots n}{6},$$

&c.

PROB. XII.

To find a general construction for any row of Table II, Prop. XII.

Let $a, b, c, d, e, \&c.$ denote the numbers in the n^{th} row, then

Row						
1	1					
2	1	1				
3	2	3	1			
4	6	11	6	1		
n	a	b	c	d	e	
n	a	b	c	d	e	&c.
1	1	1	1	1	1	

Here by the construction of the table, $na + 0 = a$, $nb + a = b$, $nc + b = c$, $nd + c = d$, &c. that is, $na = a + a$, and $nb + a = b + b$, $nc + b = c + c$, &c. whence

$a = \overline{n-1} \cdot a$, $b = \overline{n-1} \cdot b + a$, $c = \overline{n-1} \cdot c + b$, &c. but the integrals being not to be had this way, we must try another method.

Otherwise,

Let $a, b, c, d, \&c.$ denote the numbers in a contrary order, from right to left.

Row							
1	1						
2	1	1					
3	1	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
n	f	e	d	c	b	a	
n	g	f	e	d	c	b	a
1	1	1	1	1	1	1	1

C c

Here

Here, from the construction, $n \times 0 + a = a$, $na + b = b$, $nb + c = c$, $nc + d = d$, &c. Whence $a = 0$, and $a = 1$; and $n + b = b = b + b$, and $b = n$, $c = nb$, $d = nc$, $e = nd$, $f = ne$, &c. Therefore the integrals are as follows :

$$\begin{aligned}
 b &= \frac{n \cdot n}{2} \\
 c &= \frac{n \cdot n}{2} n = \frac{n+2}{2} \times \frac{n \cdot n}{2} = \frac{n \cdot n \cdot n}{2} + \frac{2 \cdot n \cdot n}{2} \\
 \text{and } c &= \frac{n \cdot n \cdot n}{2 \cdot 4} + \frac{2 \cdot n \cdot n \cdot n}{2 \cdot 3} \\
 d = nc &= \frac{n+4}{4} \times \frac{n \cdot n \cdot n}{2 \cdot 4} + \frac{n+3}{3} \times \frac{2 \cdot n \cdot n \cdot n}{2 \cdot 3} \\
 &= \frac{n \cdot n \cdot n}{2 \cdot 4} + \frac{20 \cdot n \cdot n \cdot n}{2 \cdot 3 \cdot 4} + \frac{n \cdot n \cdot n}{3} \\
 d &= \frac{n \cdot n \cdot n}{2 \cdot 4 \cdot 6} + \frac{n \cdot n \cdot n}{2 \cdot 3} + \frac{n \cdot n \cdot n}{4} \quad \text{In like manner,} \\
 e &= \frac{n \cdot n \cdot n}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{n \cdot n \cdot n}{4 \cdot 6} + \frac{13 \cdot n \cdot n \cdot n}{3 \cdot 4 \cdot 6} + \frac{n \cdot n \cdot n}{5} \\
 f &= \frac{n \cdot n \cdot n}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \frac{n \cdot n \cdot n}{2 \cdot 3 \cdot 4 \cdot 6} + \frac{34 \cdot n \cdot n \cdot n}{3 \cdot 4 \cdot 6 \cdot 8} + \frac{11 \cdot n \cdot n \cdot n}{3 \cdot 4 \cdot 5} + \frac{n \cdot n \cdot n}{6}, \\
 &\text{\&c.}
 \end{aligned}$$

PROB. XIII.

To find any term in the series $1, \frac{1}{2}a, \frac{2}{3}b, \frac{3}{4}c, \frac{4}{5}d, \frac{5}{6}e$, &c. where a, b, c , &c. are the foregoing terms.

Let v be any term required, and let z be the place of the next term v ; then by construction of the series $v = \frac{z-1}{z} v$. Then $z v$ or $zv + zv = z-1 \cdot v$, or $zv + v = 0$, put $z-1 = x$, then $z = x+1$, and $x = z-1$. Whence $xv + v + v = 0$, that is, $v + xv = 0$.

Assume

Assume $v = A + \frac{B}{x} + \frac{C}{xx} + \frac{D}{xxx} \&c.$

Then $v = -\frac{Bx}{xx} - \frac{2Cx}{xxx} - \frac{3Dx}{xxx} \&c.$

And $v + xv =$

$$\left. \begin{aligned} &A + \frac{B}{x} + \frac{C}{xx} + \frac{D}{xxx} \&c. \\ &-\frac{Bx}{x} - \frac{2Cx}{xx} - \frac{3Dx}{xxx} - \&c. \end{aligned} \right\} = 0.$$

Whence equating the coefficients, $A=0$, $B-Bx=0$, that is, $B-B=0$, or $B=B$. $C-2Cx=0$, or $C=0$; $D-3Dx=0$, and $D=0$, &c. Whence $v = \frac{B}{x}$.

Now to determine B , we may observe, that when $z=2$, $x=1$, $v = \frac{1}{2}a = \frac{1}{2}$, and $v=1$. Whence we have $v = \frac{B}{x}$, that is, $1 = \frac{B}{1}$, or $B=1$. Therefore any term at the place $z-1$ or x will be $\frac{1}{z-1}$ or $\frac{1}{x}$. And this is also evident from the nature of the series.

Note, If we had prosecuted the question with z instead of x , we had got

$$v = \frac{1}{z} + \frac{1}{z} \alpha + \frac{2}{z} \beta + \frac{3}{z} \gamma + \frac{4}{z} \delta \&c. \text{ which is}$$

the same in value as $\frac{1}{z-1}$; where α, β, γ , &c. are the preceeding terms.

S C H O L I U M.

In such questions as this, you must carefully mind whether z denotes the place of v , or v , &c; so that your calculation be rightly made, which must be such as always to agree with the equation defining

ing the terms of the given series. And how each term is to be expressed by z , will be found by trying particular cases.

And the same caution must be observed, if you are seeking the sum of any series, where z denotes the place of $s, s, \&c.$ All this is necessary to prevent mistakes.

PROB. XIV.

To find any term in the series $\frac{2}{1}, \frac{4}{3}a, \frac{6}{5}b, \frac{8}{7}c, \frac{10}{9}d, \frac{12}{11}e, \&c.$ where $a, b, c, \&c.$ are the foregoing terms.

Let t be any term, and z the place of t , then will $\frac{2z}{2z-1} t = t$, put $x=2z-1$, then $\frac{x+1}{x} t = t$. Here if I put $t = A + \frac{B}{x} + \frac{C}{xx} \&c.$ nothing will be obtained. Therefore I square the equation, and have $\frac{x+1}{xx} t t = t t$. Put $v = tt$. Then $xx+2x+1 \cdot v = xxv = xxv + xxv$, whence $2xv + v - xxv = 0$.

$$\text{Assume } v = Ax + B + \frac{C}{x} + \frac{D}{xx} + \frac{E}{xxx} \&c.$$

$$\text{Then } v = Ax * - \frac{Cx}{xx} - \frac{2Dx}{xxx} \&c.$$

$$xxv = xAx * - \frac{x^2Cx}{xx} - \frac{2xDxx}{xxx} \&c.$$

$$2xv = 2Ax + 2Bx + 2C + \frac{2Dx}{xx} + \frac{2Ex}{xxx} \&c.$$

These being reduced to a proper form, we shall have

$$2xv + v - xxv = 0 =$$

$$\left. \begin{aligned}
 &+ 2Axx + 2Bx + 2C + \frac{2D}{x} + \frac{2E}{xx} + \frac{2F}{xxx} \\
 &\quad - \frac{2Dx}{xx} - \frac{4Ex}{xxx} \\
 &+ Ax + B + \frac{C}{x} + \frac{D}{xx} + \frac{E}{xxx} \\
 &- xAx + Cx - \frac{Cx^2}{x} + \frac{Cx^3}{xx} + \frac{8Dx^3}{xxx} \\
 &\quad + \frac{2Dx}{x} - \frac{6Dx^2}{xx} - \frac{15Ex^2}{xxx} \\
 &\quad + \frac{3Ex}{xx} + \frac{4Fx}{xxx}
 \end{aligned} \right\} = 0.$$

Then equating the coefficients, $A=A$, $B=-\frac{1}{2}A$, $C=\frac{1}{6}A$, $D=\frac{1}{16}A$, $E=\frac{1}{128}A = .08594 A$, $F = .1757 A$, &c. And therefore

$$v \text{ or } u = A \times : x - \frac{1}{2} + \frac{1}{8x} + \frac{1}{16xx} + \frac{.086}{xxx} + \frac{.1757}{xxx} \&c.$$

Now to find the value of A , compute the 5th term, in the given series, which is 4.063493, whose square is 16.511973, and put $z=6$, and $x=11$ in the value of v , and we have $16.511973 = A \times 11 - \frac{1}{2}$

$$+ \frac{1}{8.11} + \frac{1}{16.11.13} \&c. = A \times 10.511710; \text{ whence}$$

$A = 1.570796$, which is half the circumference of a circle whose diameter is 1. And A being known, the square of any other term may be found, by the above series.

PROB. XV.

To find any term in the series $\frac{1.1}{2.2}$, $\frac{3.3}{4.4} a$, $\frac{5.5}{6.6} b$, $\frac{7.7}{8.8} c$, $\frac{9.9}{10.10} d$,

$$\frac{11.11}{11.12} e, \&c.$$

Let v be any term, and z the place of v ; then will $\frac{2z-1}{2z} v = v$.

D d

Put

Put $x=2z$, then $\frac{x-1}{xx}v=v$, which reduced is $2xv-v+xxv=0$.

Assume $v = \frac{A}{x} + \frac{B}{xx} + \frac{C}{x^3} + \frac{D}{x^4} \&c.$ whence we shall find
 $2xv-v+xxv =$

$$\left. \begin{array}{l} 2A + \frac{2B}{x} + \frac{2C}{xx} + \frac{2D}{x^3} \&c. \\ - \frac{A}{x} - \frac{B}{xx} - \frac{C}{x^3} \\ - Ax + \frac{Ax^2}{x} - \frac{Ax^3}{xx} + \frac{Ax^4}{x^3} \\ - \frac{2Bx}{x} + \frac{3Bx^2}{xx} - \frac{4Bx^3}{x^3} \\ - \frac{3Cx}{xx} + \frac{6Cx^2}{x^3} \\ - \frac{4Dx}{x^3} \end{array} \right\} = 0.$$

$\&c.$

Then equating the coefficients $A=A$; $B = \frac{4A-A}{2} = \frac{3}{2}A$;

$C = \frac{11B-8A}{4} = 2.125A$; $D = \frac{23C-32B+16A}{6} = 2.812A$;

$E = \frac{19D-80C+80B-32A}{8} = -3.6A$; $\&c.$

Whence $v = A \times \frac{1}{x} + \frac{1.5}{xx} + \frac{2.125}{x^3} + \frac{2.81}{x^4} - \frac{3.6}{x^5} \&c.$

Now to find A , compute the 9th term .034398; and put $z=10$,
 $x=20$; and you will find .034398 $= A \times .054035$; Whence

$A = .636619$, which is $= \frac{2}{3.1416}$; whence any other term will be
 had. It is evident, when x is infinite, the last term of the series will
 be 0.

Hence any sum of the series $\frac{1}{2}$, $\frac{3}{4}a$, $\frac{5}{6}b$, $\frac{7}{8}c$, $\frac{9}{10}d$, $\&c.$ may be
 found. For find the corresponding term of the former series, and ex-
 tract its square root, and you have the term required.

PROB. XVI.

Let t be any term of a series, and z the place of any term, as t or t_1 ; and suppose the several terms of the series to be determined by this equation, $\frac{z+n}{z} t = t_1$; to find any term in it.

Since $\frac{z+n}{z} t = t = t + t$, then $zt + nt = zt + zt$, and $nt = zt$.

Assume $t = Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3} \&c.$

$$\begin{aligned} \text{Then } t &= A \times : nz^{n-1} + \frac{nn}{2} z^{n-2} + \frac{nnn}{2.3} z^{n-3} \\ &+ B \times : \frac{n}{1} z^{n-2} + \frac{nn}{2} z^{n-3} \\ &+ C : \times \frac{n}{2} z^{n-3} \&c. \end{aligned}$$

Putting $z = 1$, and $\frac{n}{1}, \frac{n}{2}, \frac{n}{3}, \&c.$ for $n-1, n-2, n-3, \&c.$

Then $nt = nAz^n + nBz^{n-1} + nCz^{n-2} + nDz^{n-3} \&c.$

$$\begin{aligned} = & \left| \begin{aligned} & nAz^n + \frac{nnA}{2} z^{n-1} + \frac{nnn}{2.3} Az^{n-2} + \frac{n\dots n}{2.3.4} Az^{n-3} \\ & + \frac{n}{1} Bz^{n-1} + \frac{nnB}{2} z^{n-2} + \frac{nnn}{2.3} Bz^{n-3} \\ & + \frac{n}{2} Cz^{n-2} + \frac{nn}{2} Cz^{n-3} \\ & + \frac{n}{3} Dz^{n-3} \end{aligned} \right. \end{aligned}$$

Then equating the coefficients, $A=A$, $nB = \frac{n \cdot n-1}{2} B + \frac{n \cdot n-1}{2} A$,

and $B = \frac{n \cdot n-1}{2} A$. Likewise $2C = \frac{n \cdot n-1 \cdot n-2}{2.3} A +$

$\frac{n-1 \cdot n-2}{2} B.$

3 D

$$3D = \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{2 \cdot 3 \cdot 4} A + \frac{n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3} B + \frac{n-2 \cdot n-3}{1 \cdot 2} C. \text{ \&c. Whence}$$

$$\begin{aligned} t = A z^n &+ \frac{n \cdot n-1}{2} A z^{n-1} \\ &+ \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} A + \frac{n-1}{2} \cdot \frac{n-2}{2} B \times : z^{n-2} \\ &+ \frac{n}{3} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} A + \frac{n-1}{3} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} B + \frac{n-2}{3} \cdot \frac{n-3}{2} C \times : z^{n-3} \\ &+ \left. \begin{aligned} &+ \frac{n}{4} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} A \\ &+ \frac{n-1}{4} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \frac{n-4}{4} B \\ &+ \frac{n-2}{4} \cdot \frac{n-3}{2} \cdot \frac{n-4}{3} C \\ &+ \frac{n-3}{4} \cdot \frac{n-4}{2} D \end{aligned} \right\} z^{n-4} \cdot \text{\&c.} \end{aligned}$$

And the coefficient A may be determined, by seeking any term, by the given series, and also by the series assumed; which being put equal will determine A . This series is of service when n is small and z great.

Otherwise,

The equation of the series is $\frac{z+n}{z} t = t$; and $nt = zt$.

Assume $t = A z + B z z + C z z z + D z \dots z \text{ \&c.}$

Then $t = z A + 2 z B z + 3 z C z z \text{ \&c.}$

And $z t = z A z + 2 z B z \times z + 3 z C \times z z \text{ \&c.}$

That is $z t = z A z + 2 z B z z + 3 z C z \dots z \text{ \&c.}$

$+ 2 z^2 B z + 2.3 z^2 C z z + \text{\&c.}$

Whence

Whence

$$\begin{array}{l}
 nt \quad \left| \begin{array}{l} nAz + nBz \underset{-1}{z} + nCz \dots \underset{-2}{z} + nDz \dots \underset{-3}{z} \&c. \\ \\ \\ \end{array} \right. \\
 = \quad \left| \begin{array}{l} \\ \\ \\ \end{array} \right. \\
 zt \quad \left| \begin{array}{l} zAz + 2zBz \underset{-1}{z} + 3zCz \dots \underset{-2}{z} + 4zDz \dots \underset{-3}{z} \&c. \\ \\ \\ \end{array} \right. \\
 \quad \left| \begin{array}{l} \\ \\ \\ \end{array} \right. \\
 \quad \left| \begin{array}{l} + 2z^2Bz + 2.3z^2Cz \underset{-1}{z} + 3.4z^2Dz \dots \underset{-2}{z} + 4.5z^2Ez \dots \underset{-3}{z} \&c. \end{array} \right.
 \end{array}$$

Hence comparing the coefficients, where $z=1$, we have $nA=A+2B$, and $B = \frac{n-1}{2} A$, $C = \frac{n-2}{2.3} B$, $D = \frac{n-3}{3.4} C$, $E = \frac{n-4}{4.5} D$, &c. Whence

$$t = Az + \frac{n-1}{2} P \underset{-1}{z} + \frac{n-2}{2.3} Q \underset{-2}{z} + \frac{n-3}{3.4} R \underset{-3}{z} + \frac{n-4}{4.5} S \underset{-4}{z} \&c.$$

P, Q, R, &c. being the foregoing terms. And A may be determined by some given term of the series, as before.

This series is serviceable for interpolation, when n is an affirmative whole number, or z a small number.

Or when z is not 1, we shall find

$$t = Az + \frac{n-z}{2z^2} P \underset{-1}{z} + \frac{n-2z}{2.3z^2} Q \underset{-2}{z} + \frac{n-3z}{3.4z^2} R \underset{-3}{z} + \frac{n-4z}{4.5z^2} S \underset{-4}{z} \&c.$$

PROB. XVII.

If the several terms of a series be defined by this equation, $\frac{xx+r}{xx} v = \underset{1}{v}$; where $v, \underset{1}{v}$, are two succeeding terms in the series: to find any term therein.

Here $xxv + rv = xx \underset{1}{v} = xxv + xxv$, and $rv = xxv$;

E c

Assume

Assume $v = A + \frac{B}{x} + \frac{C}{xx} + \frac{D}{x^3} + \frac{E}{x^4} \&c.$ whence

$$v = -\frac{Bx}{xx} + \frac{Bx^2}{x^3} - \frac{Bx^3}{x^4} \&c.$$

$$-\frac{2Cx}{x^3} + \frac{3Cx^2}{x^4}$$

$$-\frac{3Dx}{x^4}$$

$$\&c.$$

Then $rv = rA + \frac{rB}{x} + \frac{rC}{xx} + \frac{rD}{x^3} + \frac{rE}{x^4} \&c.$

$$xxv = -Bx + \frac{Bxx}{x} - \frac{Bx^3}{xx} + \frac{Bx^4}{x^3} - \frac{Bx^5}{x^4} \&c.$$

$$-\frac{2Cx}{x} + \frac{3Cxx}{xx} + \frac{4Cx^3}{x^3} + \frac{5Cx^4}{x^4}$$

$$-\frac{3Dx}{xx} + \frac{6Dxx}{x^3} - \frac{10Dx^3}{x^4}$$

$$-\frac{4Ex}{x^3} + \frac{10Exx}{x^4}$$

$$-\frac{5Fx}{x^4} \&c.$$

Then equating the respective coefficients, we have

$$B = -\frac{rA}{x}, \quad C = \frac{xx-r}{2x} B = -\frac{xx-r}{2xx} rA.$$

$$D = \frac{3xx-r \cdot C - Bx^3}{3x}, \quad E = \frac{6xx-r \cdot D - 4Cx^3 + Bx^4}{4x},$$

$$F = \frac{10xx-r \cdot E - 10Dx^3 + 5Cx^4 - Bx^5}{5x},$$

$$G = \frac{15xx-r \cdot F - 20Ex^3 + 15Dx^4 - 6Cx^5 + Bx^6}{6x}, \&c.$$

where the numeral coefficients, are the uncia of the several powers of a binomial.

Otherwise,

Otherwise,

$$\text{Assume } v = A + \frac{B}{x} + \frac{C}{xx} + \frac{D}{xxx} + \frac{E}{x \dots x} \quad \&c.$$

$$\text{Then } v = -\frac{Bx}{xx} - \frac{2Cx}{xxx} - \frac{3Dx}{x \dots x} - \frac{4Ex}{x \dots x} \quad \&c.$$

$$\text{And } xxv = -\frac{Bxx^2}{xx} - \frac{2Cx^2x}{xxx} - \frac{3Dx^2x}{x \dots x} - \frac{4Ex^2x}{x \dots x} \quad \&c.$$

The value of xxv being reduced to a proper form, we shall have
 $rv = xxv$, that is,

$$\begin{aligned} & rA + \frac{rB}{x} + \frac{rC}{xx} + \frac{rD}{xxx} + \frac{rE}{x \dots x} + \frac{rF}{x \dots x} \quad \&c. \\ &= -Bx + \frac{Bxx}{x} - \frac{Bxx^3}{xx} - \frac{2.4Cx^3}{xxx} - \frac{3.9Dx^3}{x \dots x} - \frac{4.16Ex^3}{x \dots x} \quad \&c. \\ &\quad - \frac{2Cx}{x} + \frac{2.3Cx^2}{xx} + \frac{3.5Dx^2}{xxx} + \frac{4.7Ex^2}{x \dots x} + \frac{5.9Fx^2}{x \dots x} \\ &\quad - \frac{3Dx}{xx} - \frac{4Ex}{xxx} - \frac{5Fx}{x \dots x} - \frac{6Gx}{x \dots x} \end{aligned}$$

And equating the coefficients, $B = -\frac{rA}{x}$, $C = \frac{\overline{xx-r}}{2x} \cdot B$,

$$D = \frac{2.3 \overline{xx-r} \cdot C - Bx^3}{3x}, \quad E = \frac{3.5 \overline{xx-r} \cdot D - 2.4Cx^3}{4x},$$

$$F = \frac{4.7 \overline{xx-r} \cdot E - 3.9Dx^3}{5x}, \quad G = \frac{5.9 \overline{xx-r} \cdot F - 4.16Ex^3}{6x}, \quad \&c.$$

EXAMPLE.

To find the $\overline{z-1}^{th}$ term of the series 1, $\frac{2.4}{3.3}A$, $\frac{4.6}{5.5}B$, $\frac{6.8}{7.7}C$,
 $\frac{8.10}{9.9}D$, $\frac{10.12}{11.11}E$ &c.

Let v be the $\overline{z-1}^{th}$ term, then by the series, the z^{th} term

or $v = \frac{2z-2 \cdot 2z}{2z-1} v$. Put $x=2z-1$, then $\frac{x x-1}{x x} v = v$, which compared with this problem, we have, $r=-1$, $x=2z=2$, from whence, by the first method, we get, $A=A$, $B=\frac{1}{2}A$, $C=\frac{5}{8}A$, $D=\frac{11}{16}A$, $E=.648 A$, $F=.557 A$, $G=.608 A$, &c.

Therefore $v = A \times : 1 + \frac{1}{2x} + \frac{5}{8xx} + \frac{11}{16x^3} + \frac{.648}{x^4} + \frac{.557}{x^5} + \frac{.61}{x^6} \&c.$

Or $v = A \times : 1 + \frac{1}{2x} + \frac{5}{4x} P + \frac{11}{10x} Q + \frac{.942}{x} R + \frac{.86}{x} S + \frac{1.1}{x} T$,
&c : P, Q, R, &c. being the preceeding terms.

Now to determine A, compute any term at a good distance from the beginning, as the 10th term, which is .805272, and put $z=11$, $x=21$, then $.805272 = A \times : 1 + \frac{.5}{21} + \frac{5}{84} P + \frac{11}{10.21} Q$, &c. = $A \times 1.025304$; and therefore $A=.785398$, which is the area of a circle whose diameter is 1. And A being known, any other term may be found by the series above.

When x is infinite, it is evident, the last term of the series, or the product of all these numbers, $1 \times \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \&c. \text{ ad infinitum}$, is = .785398.

Or thus by the second method,

Let z be the place of the term v , then $\frac{2z \cdot 2z+2}{2z+1} v = v$; put $x = 2z+1$, and then $\frac{x x-1}{x x} v = v$; hence we have $r=-1$, $x=2z=2$. And therefore $B=\frac{1}{2}A$, $C=\frac{5}{8}A$, $D=1.9375 A$, $E=9.76 A$, $F=68.45 A$, $G=616 A$, &c. Whence

$v = A \times : 1 + \frac{1}{2x} + \frac{5}{8xx} + \frac{1.9375}{xxx} + \frac{9.76}{x \dots x} + \frac{68.45}{x \dots x} + \frac{616}{x \dots x} \&c.$

Let $z=10$, then $x=21$, and we shall have $.805272 = A \times : 1 + \frac{1}{2.21} + \frac{5}{8.21.23} + \frac{1.9375}{21.23.25} \&c. = A \times 1.025304$, and and $A=.785398$, the same as before.

PROB. XVIII.

If the terms of a series be defined by this equation, $\frac{z z}{z z + r} t = t$; to find any term in the series, expressed by $z, z, \&c.$

From the equation of the terms $z z t = \overline{z z + r} \times t = \overline{z z + r} \cdot t + \overline{z z + r} \cdot t$,
and $r t + z z t + r t = 0 = r t + z z t$.

$$\text{Assume } t = A + \frac{B}{z} + \frac{C}{z z} + \frac{D}{z z z} \&c.$$

$$\text{Then } t = - \frac{B z}{z z} - \frac{2 C z}{z z z} - \frac{3 D z}{z \dots z} \&c.$$

$$\text{And } z z t = - \frac{B z z}{z} - \frac{2 C z z}{z z} - \frac{3 D z z}{z \dots z} \&c.$$

$$= - B z + \frac{B z z - 2 C z}{z} + \frac{4 C z z - 3 D z}{z z} \&c.$$

Whence $r t + z z t =$

$$\left. \begin{aligned} & r A + \frac{r B}{z} + \frac{r C}{z z} + \frac{r D}{z z z} \&c. \\ & - B z + \frac{B z z - 2 C z}{z} + \frac{4 C z z - 3 D z}{z z} + \frac{9 D z z - 4 E z}{z z z} \&c. \end{aligned} \right\} = 0.$$

Comparing the terms, $B = \frac{r A}{z}$, $C = \frac{r + z z}{2 z} B$, $D = \frac{r + 4 z z}{3 z} C$,

$E = \frac{r + 9 z z}{4 z} D$. Whence

$$t = A + \frac{r A}{z z} + \frac{r + z z}{2 z z z} B + \frac{r + 4 z z}{3 z z \dots z} C + \frac{r + 9 z z}{4 z z \dots z} D \&c.$$

F f

Or

$$\text{Or } t = A + \frac{r}{zz} P + \frac{r+zz}{2zz} Q + \frac{r+4zz}{3zz} R + \frac{r+9zz}{4zz} S + \frac{r+16zz}{5zz} T; \text{ \&c. where } P, Q, R, \text{ \&c. are the preceeding terms.}$$

And the coefficient A may be determined, by finding any term by both the series, and putting them equal; as in the former Problems.

EXAMPLE.

In the series, $1, \frac{4}{3} A, \frac{16}{15} B, \frac{36}{35} C, \frac{64}{63} D, \frac{100}{99} E, \text{ \&c.}$ to find any term t .

Let v be the place of t , then by the progression of the series, $t = \frac{4vv}{4vv-1} t$. Put $z=2v$, then $t = \frac{zz}{zz-1} t$.

Compute the 10th term, 1.530017 from the given series; then $v=10$, $z=20$, and $r=-1$. $z=2$, which substituted in the equation

$$t = A + \frac{r}{zz} P + \frac{r+zz}{2zz} Q, \text{ \&c. we have}$$

$$1.530017 = A \times 1 - \frac{1}{2.20} P + \frac{3}{4.22} Q + \frac{15}{6.24} R + \frac{35}{8.26} S + \frac{63}{10.28} T, \text{ \&c.} = A \times .974039; \text{ whence } A = 1.570796, \text{ the semicircumference of a circle. And } A \text{ being known any other term } t, \text{ at the place } v, \text{ will be had by this series,}$$

$$t = A \times 1 - \frac{1}{2z} P + \frac{3}{4z} Q + \frac{15}{6z} R + \frac{35}{8z} S, \text{ \&c. where } z = 2v.$$

PROB. XIX.

If a series is generated by this equation, $zz+az+bx \times t = \overline{zz+cz} \times t$; to find any term therein.

The equation reduced is $\overline{az-cz} \times t + bt = zzt + czt$. Put $a-c=n$, and assume $t = Az^n + Bz^{n-1} + Cz^{n-2} \text{ \&c.}$ then t will be as in Prob. XVII, and the equation will be expressed thus:

$$+ nzt$$

$$\begin{array}{l}
 +nzt \quad n A z^{n+1} + n B z^n + n C z^{n-1} + n D z^{n-2} \&c. \\
 +bt \quad \quad \quad + b A z^n + b B z^{n-1} + b C z^{n-2} \\
 = \quad \quad \quad = \\
 zzt \quad \quad \quad \frac{n \cdot n}{2} A z^n + \frac{n \cdot n \cdot n}{6} A z^{n-1} + \frac{n \cdot n \cdot n}{24} A z^{n-2} \\
 \quad \quad \quad + \frac{n \cdot n}{2} B z^{n-1} + \frac{n \cdot n \cdot n}{6} B z^{n-2} \\
 \quad \quad \quad + \frac{n \cdot n}{2} C z^{n-2} + \frac{n \cdot n}{3} D z^{n-2} \\
 +czt \quad \quad \quad + c n A z^n + \frac{n \cdot n}{2} c A z^{n-1} + \frac{n \cdot n \cdot n}{6} c A z^{n-2} \\
 \quad \quad \quad + \frac{n \cdot n}{2} c B z^{n-1} + \frac{n \cdot n \cdot n}{2} c B z^{n-2} \\
 \quad \quad \quad + \frac{n \cdot n}{2} c C z^{n-2} \\
 \quad \quad \quad \&c.
 \end{array}$$

Whence $A = A$. $B = \frac{n \cdot n - 1}{2} + c n - b \times$ into A .

$$\begin{aligned}
 C &= \frac{n \cdot n - 1 \cdot n - 2}{6} + \frac{n \cdot n - 1}{2} c \times \text{into } \frac{1}{2} A. \\
 &+ \frac{n - 1 \cdot n - 2}{2} + n - 1 \cdot c - b \times \text{into } \frac{1}{2} B.
 \end{aligned}$$

$$\begin{aligned}
 D &= \frac{n \cdot n - 1 \cdot n - 2 \cdot n - 3}{24} + \frac{n \cdot n - 1 \cdot n - 2}{6} c \times \text{into } \frac{1}{3} A. \\
 &+ \frac{n - 1 \cdot n - 2 \cdot n - 3}{6} + \frac{n - 1 \cdot n - 2}{2} c \times \text{into } \frac{1}{3} B. \\
 &+ \frac{n - 2 \cdot n - 3}{2} + n - 2 \cdot c - b \times \text{into } \frac{1}{3} C. \\
 &\&c.
 \end{aligned}$$

where A being indetermined, may be found as in the other Problems.

P R O B.

PROB. XX.

To find the product of any number of the quantities, $a \times \overline{a+r} \times \overline{a+2r} \times \overline{a+3r} \times \overline{a+4r} \times \dots$ &c.

It is certain, if the log. of that product can be found, the number answering to it will be known. But the sum of the logarithms of the factors is the log. of that product. Therefore let v be the last term, and put

$$s = l:a + l:\overline{a+r} + l:\overline{a+2r} \dots + l:v$$

Then the next following term is $l:v+r$, where $v=r$, that is,

$$s = l:\overline{v+r} = l:v + \frac{r}{v} - \frac{rr}{2vv} + \frac{r^3}{3v^3} - \frac{r^4}{4v^4} + \dots \text{ by the nature of logarithms.}$$

$$\text{Assume } s = Avl:v + Bl:v + Cv + \frac{D}{v} + \frac{E}{v^2} + \frac{F}{v^3} \dots$$

Then by Exam. 18 and 20. Prop. XI.

$$\begin{aligned} s = AvV + Av + \frac{Av^2}{2v} - \frac{Av^3}{6v^2} + \frac{Av^4}{12v^3} - \frac{Av^5}{20v^4} \dots \\ + \frac{Bv}{v} - \frac{Bvv}{2vv} + \frac{Bv^3}{3v^3} - \frac{Bv^4}{4v^4} \\ + Cv \quad * \quad - \frac{Dv}{vv} + \frac{Dvv}{v^3} - \frac{Dv^3}{v^4} \\ - \frac{2Ev}{v^3} + \frac{3Evv}{v^4} \\ - \frac{3Fv}{v^4} \dots \end{aligned}$$

$$= V \quad * \quad + \frac{r}{v} - \frac{rr}{2vv} + \frac{r^3}{3v^3} - \frac{r^4}{4v^4} \dots$$

where $V = \text{hyp. log: } v$, and the rest, hyp. logs.

Now equating the coefficients $Av=1$, and $A = \frac{1}{v} = \frac{1}{r}$. $B = \frac{1}{2}$,

$$C = -\frac{1}{r}, D = \frac{1}{12}r, E=0, F = -\frac{1}{360}r^3, G=0, H = -$$

$$\frac{13r^5}{1260}, \dots \text{ Hence}$$

$$s =$$

$$s = \frac{v + \frac{1}{2}r}{r} \log. v - \frac{v}{r} + \frac{r}{12v} - \frac{r^3}{360v^3} - \frac{13r^5}{1260v^5} \&c.$$

But when $s = \log. a$, then $\frac{v + \frac{1}{2}r}{r} \log. v - \frac{v}{r} \&c. = \frac{a + \frac{1}{2}r}{r} \log. a - \frac{a}{r} \&c.$ Therefore $s - l : a =$ difference of these series. Or $s =$ that difference $+ l : a$, that is,

$$s = \frac{v + \frac{1}{2}r}{r} l : v - \frac{v}{r} + \frac{r}{12v} - \frac{r^3}{360v^3} - \frac{13r^5}{1260v^5} \&c. \\ - \frac{a - \frac{1}{2}r}{r} l : a + \frac{a}{r} - \frac{r}{12a} + \frac{r^3}{360a^3} + \frac{13r^5}{1260a^5} \&c.$$

E X A M P L E.

To find the product of all the natural numbers from 1 to 100.

The product of 1, 2, 3, &c. to 9 is 362880, whose common log. is 5.5597630. Then $a=10$, $v=100$, $r=1$, and hyp. log. $a=2.3025851$, and hyp. log. $v=4.6051702$, and $v + \frac{1}{2}r = 100.5$, also $a - \frac{1}{2}r = 9.5$;

then $\frac{v + \frac{1}{2}r}{r} l : v = 462.81960$ $-\frac{v}{r} = -100.00000$ $+ \frac{r}{12v} = + 83$ + 462.82043 + 362.82043 - 11.88289 + 350.93754	$-\frac{a - \frac{1}{2}r}{r} l : a = -21.87456$ $+\frac{a}{r} = + 10.00000$ $-\frac{r}{12a} = - 833$ $+\frac{r^3}{360a^3} = + 0$ - 21.88289 - 11.88289
---	--

the hyp. log. of the product.

And $350.93754 \times .4342945 = 152.41024$ the vulgar log.
 To which add the log. 362880 = $\frac{5.55976}{157.97000}$

the log. of the

product of all the natural numbers to 100, and that product is 933256 (152), the number 933256, with 152 figures or cyphers annexed.

Or it may be computed thus,

Since $S = \frac{v+\frac{1}{2}r}{r} V - \frac{v}{r} + \frac{r}{12v} - \frac{r^3}{360v^3} \&c.$ the sum of 100 terms.

And $s = \frac{a+\frac{1}{2}r}{r} A - \frac{a}{r} + \frac{r}{12a} - \frac{r^3}{360a^3} \&c.$ the sum of 10 terms.

Therefore $S - s = \frac{v+\frac{1}{2}r}{r} V - \frac{v}{r} + \frac{r}{12v} \&c. - \frac{a+\frac{1}{2}r}{r} A + \frac{a}{r} - \frac{r}{12a} \&c.$

And $S = \frac{v+\frac{1}{2}r}{r} V - \frac{v}{r} + \frac{r}{12v} - \frac{r^3}{360v^3} \&c.$
 $- \frac{a+\frac{1}{2}r}{r} A + \frac{a}{r} - \frac{r}{12a} + \frac{r^3}{360a^3} \&c. + s.$

Where $A = \text{hyp. log. } a$, and $s = \text{sum of the logs. of } 1, 2, 3, \&c. \text{ to } 10$; or the log. 3628800. = 6.559763.

PROB. XXI.

To find the product of any number of the quantities, $\frac{a}{b} \times \frac{a+r}{b+r} \times$
 $\frac{a+2r}{b+2r} \times \frac{a+3r}{b+3r} \times \frac{a+4r}{b+4r} \&c.$

Let $\frac{v}{y}$ be the last term, and put $S = \log. \frac{a}{b} + l : \frac{a+r}{b+r} +$
 $l : \frac{a+2r}{b+2r} + l : \frac{a+3r}{b+3r} \dots + l : \frac{v}{y}.$ Then the next following term
 is $l : \frac{v+r}{y+r}$, where $v=y=r$, that is $S = l : \frac{v+r}{y+r}$. Put $y+r=x$, then
 since $a-b=v-y$, we have $v=a-b+y$, and $v+r=y+r+a-b=x+a-b$
 $=x+n$ (putting $n=a-b$); therefore $S = l : \frac{x+n}{x} = \frac{n}{x} - \frac{n^2}{2xx}$
 $+ \frac{n^3}{3x^3} - \frac{n^4}{4x^4} \&c.$

Assume $S = A l : x + \frac{B}{x} = \frac{C}{xx} + \frac{D}{x^3} + \frac{E}{x^4} \&c.$ Then
 (Ex. 18. Prop. XI.)

$S =$

$$\begin{aligned}
 S &= \frac{Ax}{x} - \frac{Ax^2}{2x^2} + \frac{Ax^3}{3x^3} + \frac{Ax^4}{4x^4} + \frac{Ax^5}{5x^5} \&c. \\
 &\quad - \frac{Bx}{xx} + \frac{Bx^2}{x^3} - \frac{Bx^3}{x^4} + \frac{Bx^4}{x^5} \\
 &\quad - \frac{2Cx}{x^3} + \frac{3Cx^2}{x^4} - \frac{4Cx^3}{x^5} \\
 &\quad - \frac{3Dx}{x^4} + \frac{6Dx^2}{x^5} \\
 &\quad - \frac{4Ex}{x^5} \&c. \\
 &= \frac{n}{x} - \frac{n^2}{2xx} + \frac{n^3}{3x^3} - \frac{n^4}{4x^4} + \frac{n^5}{5x^5} \&c.
 \end{aligned}$$

And equating the coefficients, $Ax=n$, or $A = \frac{n}{x}$, and $B = \frac{nn-Ax^2}{2x}$

$$= \frac{n-x}{2x} n. \quad C = \frac{Ax^3-n^3+3Bxx}{2.3x} = \frac{3nx-2nn-xx}{3.4x} n.$$

$$D = \frac{n^4-Ax^4}{3.4x} - \frac{Bx^2}{3} + \frac{3Cx}{3} = \frac{n-x}{3.4x} n n.$$

$$E = \frac{Ax^5-n^5}{4.5x} + \frac{Bx^3}{4} - \frac{4Cx^2}{4} + \frac{6Dx}{4}.$$

$$F = \frac{n^6-Ax^6}{5.6x} - \frac{Bx^4}{5} + \frac{5Cx^3}{5} - \frac{10Dx^2}{5} + \frac{10Ex}{5} \&c.$$

the coefficients of B, C, D, being those formed from a binomial.

Whence

$$S = AX + \frac{B}{x} + \frac{C}{xx} + \frac{D}{x^3} + \frac{E}{x^4} + \frac{F}{x^5} \&c.$$

where $X = \text{hyp. log. } x$, and S the hyp. log. of the product.

Or rather thus; Let S, X represent the common logarithms of the quantities, then

$$S = AX + .43429448 \times : \frac{B}{x} + \frac{C}{xx} + \frac{D}{x^3} + \frac{E}{x^4} + \frac{F}{x^5} \&c.$$

But when $x=b$, $S=0$. Therefore, by correction,

$$\begin{aligned}
 S &= AX + .43429448 \times : \frac{B}{x} + \frac{C}{xx} + \frac{D}{x^3} + \frac{E}{x^4} + \frac{F}{x^5} \&c. \\
 &\quad - A \times \log : b - .43429448 \times : \frac{B}{b} + \frac{C}{bb} + \frac{D}{b^3} + \frac{E}{b^4} + \frac{F}{b^5} \&c.
 \end{aligned}$$

Where

Where

$$A = \frac{n}{r}.$$

$$B = \frac{n-r}{2r} n.$$

$$C = \frac{3nr-2nn-rr}{12r} n.$$

$$D = \frac{n-r}{3.4r} n n.$$

$$E = \frac{r^4-n^4}{4.5r} n + \frac{r^3B-4rrC+6rD}{4}.$$

$$F = \frac{n^5-r^5}{5.6r} n + \frac{-r^4B+5r^3C-10r^2D+10rE}{5}.$$

$$G = \frac{r^6-n^6}{6.7r} n + \frac{r^5B-6r^4C+15r^3D-20r^2E+15rF}{6}.$$

&c.

EXAMPLE.

To find the product of 100 terms of the series, $\frac{21.24.27.30}{20.23.26.29} \&c.$

Here $b=20$, $n=1$, x or $r=3$, $l:b = 1.3010300$. Also $y=317$, $x=320$, $l:x = 2.5051500$. And $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = -\frac{1}{18}$, $D = \frac{1}{9}$, $E = \frac{1}{12}$, $F = -\frac{1}{48}$, &c.

Therefore

$$S = \frac{2.505150}{3} - \frac{.4342945}{3.320} - \frac{.4342}{18.320^2} + \frac{.43}{9.320^3} \&c.$$

$$- \frac{1.3010300}{3} + \frac{.4342}{3.320} \&c. + \frac{.4342}{18.320^2} + \frac{.434}{9.320^3} + \frac{.43}{12.320^4} \&c.$$

Which summed up is as follows:

+ .835050	- .433676
- 453	+ 7238
+ .834597	+ 60
- .426384	- 6
Log = +0.408213	- .433682
Number 2.55984	+ 7298
	- .426384
	= 2.56

So the product required is nearly
It is easy to see, that the product of all these terms, continued *ad infinitum*, will be infinite.

PROB.

P R O B. XXII.

To find the sum of any number of terms of the series, $1.2 + 2.3 + 3.4 + 4.5 + 5.6$ &c.

Let z = number of terms.
 s = sum of them.

Then by the progression of the series, the last or z^{th} term $= z z$, and the next term to be added will be $z z$, that is, $s = z z$, where $z=1$.

Consequently the integral is $s = \frac{z z z}{3}$, for when $z=1$, $s = \frac{z z z}{3}$, as it ought to be by the series. Therefore the sum of z terms $= \frac{z. z+1. z+2}{3}$.

P R O B. XXIII.

To find the sum of n terms of the series, $1.2.4 + 3.4.6 + 5.6.8 +$ &c.

Let s = the sum; the n^{th} term will be $\frac{2n-1}{1} \cdot 2n \cdot \frac{2n+2}{1}$, by the progression of the series. But $2n-1 = 2n+2-1 = \frac{2n+1}{1}$, because

$n=1$. Therefore the n^{th} term $= \frac{2n+1}{1} \times 4nn = \frac{8nnn}{1} + \frac{4nn}{1}$; and the next term will be $\frac{8nnn}{1} + \frac{4nn}{1} = s$. Then the integral

$$s = \frac{\frac{8nnn}{1}}{4n} + \frac{\frac{4nn}{1}}{3n} = \frac{2nnnn}{1} + \frac{4nnn}{3} = \frac{2n+1}{1} \times \frac{nnn}{1} = \frac{2n-2+\frac{4}{3}}{1} \times \frac{nnn}{1} = \frac{6n-2}{3} \times n. n+1. n+2.$$

P R O B. XXIV.

To find the sum of the series $\frac{10}{1.2.3.4} + \frac{14}{2.3.4.5} + \frac{18}{3.4.5.6} + \frac{22}{4.5.6.7} +$ &c.

Let s = sum of n terms, then the last term of the sum will be $\frac{4n+6}{nnnn}$, and the next term or $s = \frac{4n+6}{nnnn} = \frac{4}{nnn} + \frac{6}{n..n}$;
H h and

and the integral $s = A - \frac{2}{\underset{2 \ 3}{n \ n}} - \frac{2}{\underset{1 \ 2 \ 3}{n \ n \ n}} = A - \frac{2n+2}{\underset{1 \ 2 \ 3}{n \ n \ n}} = A -$

$$\frac{2n+4}{\underset{1 \ 2 \ 3}{n \ n \ n}} = A - \frac{2n}{\underset{1 \ 2 \ 3}{n \ n \ n}} = A - \frac{2}{\underset{1 \ 3}{n \ n}} = \frac{2}{3} - \frac{2}{\underset{1 \ 3}{n \ n}}, \text{ for when } n=1,$$

$$s = \frac{2}{3} - \frac{2}{\underset{1 \ 3}{n \ n}}; \text{ therefore } s = \frac{2}{3} - \frac{2}{n+1 \cdot n+3}, \text{ or } s = \frac{2}{3} \times$$

$$\frac{n \cdot n+4}{n+1 \cdot n+3}:$$

PROB. XXV.

To find the sum of n terms of the series $\frac{1}{1.5} + \frac{1}{3.7} + \frac{1}{5.9} + \frac{1}{9.11}$ &c.

Let $s = \text{sum}$, then the last term is $\frac{1}{2n-1 \cdot 2n+3}$; let $v=2n-1$,

and $v = 2n = 2$. Then the last term $= \frac{1}{v \underset{2}{v}} = \frac{v}{\underset{1 \ 2}{v \ v \ v}}$; and the

next term of $s = \frac{v}{\underset{1 \ 2 \ 3}{v \ v \ v}} = \frac{v+2}{\underset{1 \ 2 \ 3}{v \ v \ v}} = \frac{1}{\underset{2 \ 3}{v \ v}} + \frac{2}{\underset{1 \ 2 \ 3}{v \ v \ v}}$; and the inte-

gral is $s = A - \frac{1}{\underset{3 \cdot}{v \ v}} - \frac{1}{\underset{1 \ 2 \cdot}{v \ v \ v}} = A - \frac{1}{2} \times \frac{v+1}{\underset{1 \ 2}{v \ v}} = A - \frac{1}{2} \times \frac{v+3}{\underset{1 \ 2}{v \ v}}$

$$= A - \frac{n+1}{2n+1 \cdot 2n+3} = \frac{1}{3} - \frac{n+1}{2n+1 \cdot 2n+3}, \text{ because when } n=1,$$

$$s = \text{that quantity}; \text{ or } s = \frac{n \cdot 4n+5}{3 \cdot 2n+1 \cdot 2n+3}.$$

PROB. XXVI.

To find (s) the sum of n terms of the series $\frac{5}{1.2.3.4} + \frac{9}{3.4.5.6} + \frac{13}{5.6.7.8}$ &c.

Here the n^{th} term $= \frac{4n+1}{2n-1 \cdot 2n \cdot 2n+1 \cdot 2n+2}$, and the next
term

term $s = \frac{4n+1=4n+5}{2n+1 \cdot 2n+2 \cdot 2n+3 \cdot 2n+4}$. Put $v=2n+1$, $x=2n+2$,

then $v=2n=2=x$, and $s = \frac{x+v}{x \cdot x \cdot v \cdot v}$; whose integral $s = A -$

$\frac{1}{x \cdot v \cdot v}$, by form 12th of the table. But when $s=0$, $n=0$, $v=1$,

$x=2$. Then $0=A - \frac{1}{1 \cdot 2 \cdot 2} = A - \frac{1}{4}$, therefore $A = \frac{1}{4}$, and

$$s = \frac{1}{4} - \frac{1}{2 \cdot x \cdot v}, \text{ or } s = \frac{n \cdot 2n+3}{2 \cdot 2n+1 \cdot 2n+2}$$

PROB. XXVII.

To find (s) the sum of n terms of the series, $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{4}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$
 $+ \frac{9}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$ &c.

The n^{th} term is $\frac{n \cdot n}{3n-2 \cdot 3n-1 \cdot 3n \cdot 3n+1 \cdot 3n+2} =$

$\frac{\frac{1}{3}n}{3n-2 \cdot 3n-1 \cdot 3n+1 \cdot 3n+2}$; and the next term $s =$
 $\frac{n=n+1}{3 \cdot 3n+1 \cdot 3n+2 \cdot 3n+4 \cdot 3n+5}$.

Put $v=3n+1$, $x=3n+2$, then

$v=3n=3=x$, and $s = \frac{x+v}{18 \cdot v \cdot v \cdot x \cdot x}$, whose integral is $s = A -$

$\frac{1}{18 \cdot v \cdot x \cdot x}$, by form 12th. But when $s=0$, $n=0$, $v=1$, $x=2$, then

$0 = A - \frac{1}{18 \cdot 1 \cdot 2 \cdot 3} = A - \frac{1}{108}$; therefore $s = \frac{1}{108} -$

$$\frac{1}{54 \cdot 3n+1 \cdot 3n+2} = \frac{n \cdot n+1}{12 \cdot 3n+1 \cdot 3n+2}$$

PROB.

PROB. XXVIII.

To find the sum of x terms of the series $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \frac{1}{13.15}$ &c. which is $\frac{1}{2}$ the circumference, when diameter $= 1$.

Let $s =$ the sum, then the x^{th} term $= \frac{1}{4x-3 \cdot 4x-1}$, and the next term or $s = \frac{1}{4x+1 \cdot 4x+3}$. Put $v=4x+1$, then $v=4x=4$;

and $s = \frac{1}{v \times v+2v} = \frac{1}{v \cdot v}$. But by Cor. Prop. III. $\frac{1}{v} = \frac{1}{v}$

$+ \frac{v}{2v \cdot v} + \frac{3 \cdot v^2}{4v \cdot v} + \frac{3 \cdot 5 \cdot v^3}{8v \cdot v} + \frac{3 \cdot 5 \cdot 7 \cdot v^4}{16v \cdot v}$ &c. Whence

$s = \frac{1}{v \cdot v} + \frac{v}{2v \cdot v} + \frac{3 \cdot v^2}{4v \cdot v}$ &c. and the integral

$s = Q - \frac{1}{v \cdot v} - \frac{1}{4v \cdot v} - \frac{3 \cdot v}{3 \cdot 4v \cdot v} - \frac{3 \cdot 5 \cdot v^2}{4 \cdot 8v \cdot v}$ &c. =

$Q - \frac{1}{4v} + \frac{4}{4v} A + \frac{2 \cdot 3 \cdot 4}{6v} B + \frac{3 \cdot 5 \cdot 4}{8v} C + \frac{4 \cdot 7 \cdot 4}{10v} D + \frac{5 \cdot 9 \cdot 4}{12v} E$

$- \frac{6 \cdot 11 \cdot 4}{14v} - \frac{7 \cdot 13 \cdot 4}{16v}$ &c. To find Q , sum up 10 terms of the

given series, which call T ; then $T = Q - \frac{1}{4 \cdot 41} + \frac{4}{4 \cdot 45}$ &c.

Therefore $Q = T + \frac{1}{4 \cdot 41} + \frac{4}{4 \cdot 45} A + \frac{2 \cdot 3 \cdot 4}{6 \cdot 49} B$ &c.

And hence the sum of all the terms *ad infinitum* after the x^{th} term, is $\frac{1}{4v} + \frac{4}{4v} A + \frac{2 \cdot 3 \cdot 4}{6 \cdot v} B +$ &c.

Otherwise,

Otherwise,

Since $s = \frac{1}{4x+1 \cdot 4x+3}$; put $z=4x+2$, then $z=4x=4$. Then
 $s = \frac{1}{zz-1}$; and assume $s = Q + \frac{A}{z} + \frac{B}{zz} + \frac{C}{z^3}$ &c. Then
 $s = Q + \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3}$ &c. $= Q + \frac{A}{z+z} + \frac{B}{z+z^2} +$
 $\frac{C}{z+z^3}$ &c. These quantities being involved, the operation will be as
 follows :

$$s = Q + \frac{A}{z} + \frac{B}{zz} + \frac{C}{z^3} + \frac{D}{z^4} + \frac{E}{z^5} \text{ \&c.}$$

$$s = Q + \frac{A}{z} - \frac{Az}{zz} + \frac{Az^2}{z^3} - \frac{Az^3}{z^4} + \frac{Az^4}{z^5}$$

$$+ \frac{B}{zz} - \frac{2Bz}{z^3} + \frac{3Bz^2}{z^4} + \frac{4Bz^3}{z^5}$$

$$+ \frac{C}{z^3} - \frac{3Cz}{z^4} + \frac{6Cz^2}{z^5}$$

$$+ \frac{D}{z^4} - \frac{4Dz}{z^5}$$

$$+ \frac{E}{z^5}$$

Then taking the difference of s and s , we shall have

$$s = -\frac{Az}{zz} + \frac{Az^2}{z^3} - \frac{Az^3}{z^4} + \frac{Az^4}{z^5} \text{ \&c.}$$

$$- \frac{2Bz}{z^3} + \frac{3Bz^2}{z^4} - \frac{4Bz^3}{z^5}$$

$$- \frac{3Cz}{z^4} + \frac{6Cz^2}{z^5}$$

$$- \frac{4Dz}{z^5}$$

$$\text{But } s = \frac{1}{zz-1} = \frac{1}{zz} * + \frac{1}{z^4} * + \frac{1}{z^6} \text{ \&c.}$$

Therefore putting these two values of s equal to each other, and equating the coefficients, we have

I i

A =

$$A = -\frac{1}{z} = -\frac{1}{4},$$

$$B = \frac{\dot{A}z}{z} = -\frac{1}{2},$$

$$C = \frac{-\dot{A}z^3 + 3Bz^2 - 1}{3z} = -\frac{1}{4},$$

$$D = \frac{Az^3 - 4Bz^2 + 6Cz}{4z} = -\frac{1}{2},$$

$$E = \frac{-Az^5 + 5Bz^4 - 10Cz^3 + 10Dz^2 - 1}{5z} = +\frac{1}{4},$$

$$F = \frac{Az^5 - 6Bz^4 + 15Cz^3 - 20Dz^2 + 15Ez}{6} = -\frac{1}{2},$$

$$G = \frac{-Az^7 + 7Bz^6 - 21Cz^5 + 35Dz^4 - 35Ez^3 + 21Fz^2 - 1}{7z} = -\frac{63}{4},$$

&c.

It is evident, the numeral coefficients in each, are the unciae of a binomial.

$$\text{Therefore } s = Q - \frac{1}{4z} - \frac{1}{2z^2} - \frac{3}{4z^3} - \frac{1}{2z^4} + \frac{3}{4z^5} - \frac{1}{2z^6} \text{ \&c.}$$

where Q may be determined as before.

PROB. XXIX.

To find (s) the sum of n terms of the series $\frac{2}{1.3.3} + \frac{3}{3.5.9} +$
 $\frac{4}{5.7.27} + \frac{5}{7.9.81} \text{ \&c.}$

Here the n^{th} term is $\frac{n+1}{2n-1 \cdot 2n+1 \cdot 3^n}$, and the $\overline{n+1}^{\text{th}}$ term or

$$s = \frac{n+2}{2n+1 \cdot 2n+3 \cdot 3^{n+1}}. \text{ Let } v=2n+1, \text{ then } v+2=2n+3. \text{ And}$$

$$2n+3=v+2, \text{ and } n+2 = \frac{v+3}{2}. \text{ Therefore } s = \frac{v+3}{2 \cdot v \cdot v \cdot 3^{n+1}} =$$

$$\frac{v+1}{2 \cdot v \cdot v \cdot 3^{n+1}} = \frac{1}{3} \left| \frac{n+1}{v} \right| \times \left(\frac{1}{2v} + \frac{1}{2 \cdot v \cdot v} \right). \text{ And (by Exam. 19.}$$

Prop.

Prop. XIII.) the integral $s = \frac{r^z}{R} \times a + \frac{b}{v} + \frac{v r}{R v} P + \&c.$
 $+ \frac{c}{v v}$

Where $b = \frac{1}{2}$, $c = \frac{1}{2}$; $a, d, e, \&c. = 0$, $z = n = 1$, $r = \frac{1}{3}$, $R = -\frac{2}{3}$,

$$P = \frac{1}{2 v}, \quad \frac{v r}{R} = -1. \text{ that is } s = \frac{r^{n+1}}{R} \times \text{into } \frac{1}{2 v} -$$

$$\frac{1}{2 v v} + \&c. \quad \text{Therefore } Q = 0,$$

$$+ \frac{1}{2 v v}$$

and $S, T, \&c. = 0$, whence $s = \frac{r^{n+1}}{R} \times \frac{1}{2 v} = - \frac{1}{3^n \cdot 4 v}$
 $= - \frac{1}{3^n \cdot 4 \cdot 2n+1}$. But when $s=0$, then $n=0$, and the inte-

gral corrected is $s = \frac{1}{4} - \frac{1}{4 \cdot 3^n \cdot 2n+1}$. Or the integral may be

found by Exam. 15, which gives 2 series, where all the terms destroy one another except the first.

P R O B. XXX.

To find (s) the sum of x terms of the series, $\frac{1}{a} + \frac{8}{a^2} + \frac{27}{a^3} + \frac{64}{a^4} \&c.$

Here the x^{th} term $= \frac{x^3}{a^x}$, and the $x+1^{\text{th}}$ term or $s = \frac{x+1^3}{a^{x+1}}$,

put $x+1=v$, then $s = \frac{v^3}{a^v}$, that is (by Exam. 20. Prop. XII),

$$s = \frac{v v v}{a^v} - \frac{3 v v}{a^v} + \frac{v}{a^v} \quad (v \text{ or } x \text{ being } = 1). \quad \text{And by}$$

Exam. 14. Prop. XIII. the integral is

$s =$

$$s = - \left\{ \begin{array}{l} \frac{v v v}{a-1} - \frac{3 v v}{a-1^2} - \frac{6 v}{a-1^3} - \frac{6}{a-1^4} \\ + \frac{3 v v}{a-1} + \frac{6 v}{a-1^2} + \frac{6}{a-1^3} \\ - \frac{v}{a-1} - \frac{1}{a-1^2} \end{array} \right\} \times \frac{1}{a^{v-1}} + Q, \text{ that is,}$$

$$s = Q - \frac{v v v - 3 v v + v}{a-1 \cdot a^x} - \frac{3 v v - 6 v + 1}{a-1^2 \cdot a^x} - \frac{6 v - 6}{a-1^3 \cdot a^x} - \frac{6}{a-1^4 \cdot a^x}.$$

But when $x=0$, $v=1$, and $s=0$, therefore

$$Q = \frac{1}{a-1} + \frac{7}{a-1^2} + \frac{12}{a-1^3} + \frac{6}{a-1^4} = \frac{a^3 + 4a^2 + a}{a-1^4}.$$

Otherwise,

We had $s = \frac{x+1^3}{a^{x+1}} = \frac{x^3 + 3x^2 + 3x + 1}{a^{x+1}}$. Assume $s = Ax^3 + Bx^2 +$

$Cx + D + \frac{E}{x}$ &c. $\times \frac{1}{a^x} = A a x^3 + B a x^2 + C a x + D a + \frac{E a}{x}$ &c. $\times$

$\frac{1}{a^{x+1}}$. Then substituting $x+1$ for x , and the operation will be as follows:

$$s = A a x^3 + B a x^2 + C a x + D a + \frac{E a}{x} \times \frac{1}{a^{x+1}}.$$

$$\left. \begin{array}{l} s = Ax^3 + 3Ax^2 + 3Ax + A \\ + Bx^2 + 2Bx + B \\ + Cx + C \\ + D \end{array} \right\} \times \frac{1}{a^{x+1}}, \text{ then } s-s \text{ or}$$

$$s =$$

$$\begin{aligned}
 s = & \left. \begin{array}{l} A \\ -Aa \end{array} \right\} x^3 + \left. \begin{array}{l} 3A \\ B \\ -Ba \end{array} \right\} x^2 + \left. \begin{array}{l} 3Ax \\ 2B \\ C \\ -Ca \end{array} \right\} x + \left. \begin{array}{l} A \text{ \&c.} \\ B \\ C \\ D \\ -Da \end{array} \right\} \times \frac{1}{a^{x+1}} \\
 = & x^3 + 3x^2 + 3x + 1 : \times \frac{1}{a^{x+1}} .
 \end{aligned}$$

Equating the coefficients we have $A - Aa = 1$, and $A = \frac{1}{1-a}$,
 $B = \frac{-3a}{1-a^2}$, $C = \frac{1+a}{1-a^3} \times 3a$, $D = -\frac{a+4aa+a^3}{1-a^4}$,
 $E, F = 0$.

But when $x=0$, $s=0$, and $a^x = 1$. Therefore the integral corrected is, $s = \frac{a+4aa+a^3}{a-1^4} - \frac{x^3}{a-1 \cdot a^x} - \frac{3ax^2}{a-1^2 \cdot a^x} - \frac{1+a}{a-1^3 \cdot a^x} \times 3ax - \frac{a+4a^2+a^3}{a-1^4 \cdot a^x}$.

If $\frac{1}{a} - \frac{8}{a^2} + \frac{27}{a^3} - \frac{64}{a^4} \text{ \&c.}$ be given; sum up these two series separately, viz. $\frac{1}{a} + \frac{27}{a^3} + \frac{125}{a^5} \text{ \&c.}$ and $\frac{8}{a^2} + \frac{64}{a^4} + \frac{216}{a^6} \text{ \&c.}$ and their difference will be the sum required.

PROB. XXXI.

To find (s) the sum of n terms of the series, $Ma + Na^2 + Oa^3 + Pa^4 + \text{ \&c.}$

Let v stand for the coefficient of a , in the n^{th} term; then the n^{th} term will be $v a^n$, and the $n+1^{\text{th}}$ term, or $s = v a^{n+1}$. But

by Examp. 13. Prop. XIII. the integral of $a^{n+1} = \frac{a^{n+1}}{a-1}$; and
 integral of $\frac{a^{n+2}}{a-1} = \frac{a^{n+2}}{a-1^2}$, and int. $\frac{a^{n+3}}{a-1^2} = \frac{a^{n+3}}{a-1^3}$, &c.

K k

Whence,

Whence by Prop. IX. the integral of $v a^{n+1}$ or s , that is $s =$

$$\frac{a^{n+1} v}{a-1} - \frac{a^{n+2} v}{a-1^2} + \frac{a^{n+3} v}{a-1^3} - \frac{a^{n+4} v}{a-1^4} \&c. \text{ Hence if any}$$

of the quantities $v, v, v \&c.$ be $=0$, the series will be given in finite terms. And v may be found expressed by n , by help of the first, second, third, $\&c.$ differences; or by finding the increments by Proposition XI.

PROB. XXXII.

To find (s) any number of terms of the series $\frac{1}{n} - \frac{1}{n+d} + \frac{1}{n+2d}$
 $- \frac{1}{n+3d} + \frac{1}{n+4d} - \frac{1}{n+5d} + \&c.$

Adding two and two terms together, the series becomes

$$\frac{d}{n \cdot n+d} + \frac{d}{n+2d \cdot n+3d} + \frac{d}{n+4d \cdot n+5d} \&c.$$

The z^{th} term or $s = \frac{d}{n + 2z-2 \cdot d \times n + 2z-1 \cdot d}$; put $v = n +$

$2z-2 \cdot d$, then $2dz = v = 2d$, because $z=1$. Then

$$s = \frac{d}{v \cdot v+d} = \frac{d}{v \cdot v+\frac{1}{2}v} = \frac{d}{v v^{\frac{1}{2}}}. \text{ But by form 13th, the integral of}$$

$$\frac{1}{v v^{\frac{1}{2}}} = -\frac{1}{v v} - \frac{1}{4 v v} - \frac{1 \cdot 3 v}{3 \cdot 4 v \cdot v} - \frac{1 \cdot 3 \cdot 5 v^2}{4 \cdot 8 v \cdot v} - \&c. \text{ Whence}$$

$$s = Q - \frac{1}{2v} - \frac{d}{4vv} - \frac{3dd}{6vvv} - \frac{3 \cdot 5 d^3}{8v \cdot v} - \frac{3 \cdot 5 \cdot 7 d^4}{10 v \cdot v} - \&c.$$

But when $s=0$, $z=1$, and $v=n$, whence is had

$$Q = \frac{1}{2n} + \frac{d}{4n \cdot n+2d} + \frac{3dd}{6n \cdot n+4d} + \frac{3 \cdot 5 d^3}{8n \cdot n+6d} +$$

$$\frac{3 \cdot 5 \cdot 7 d^4}{10n \cdot n+8d} \&c.$$

And

And when v or z is infinite, Q is equal to the sum of all the series *ad infinitum*.

Or this series may be summed, by dividing it into two series, one affirmative, the other negative; and finding the sums separately, by Prob. XXXVI. following.

P R O B. XXXIII.

To find (s) the sum of $z-1$ terms of the series $\frac{r}{n+d} + \frac{rr}{n+2d} + \frac{r^2}{n+3d} + \frac{r^3}{n+4d} \&c.$

Here the z^{th} term or $s = \frac{r^z}{n+dz}$. Put $v = n+dz$, and $v = dz = d$, because $z=1$. Then $s = \frac{r^z}{v}$; and (by Examp. 15. Prop. XIII.)

the integral $s = \frac{r^z}{r-1 \cdot v} + \frac{dr}{r-1 \cdot v} A + \frac{2dr}{r-1 \cdot v} B + \frac{3dr}{r-1 \cdot v} C,$

&c. But when $z-1=0$, or $z=1$, then $v=n+d$, and $s=0$. Therefore the integral corrected is,

$$s = \frac{r^z}{r-1 \cdot v} + \frac{dr}{r-1 \cdot v} A + \frac{2dr}{r-1 \cdot v} B + \frac{3dr}{r-1 \cdot v} C \&c.$$

$$- \frac{r}{r-1 \cdot n+d} + \frac{dr}{r-1 \cdot n+2d} A + \frac{2dr}{r-1 \cdot n+3d} B + \frac{3dr}{r-1 \cdot n+4d} C \&c.$$

where $A, B, C, \&c.$ are the preceding terms in each series.

The former series is the sum of all the terms *ad infinitum* after the $z-1^{th}$ term. And the latter series is the sum of the whole *ad infinitum*, when r is less than 1.

If the latter series doth not converge fast enough; sum up 10 terms of the given series, by which the integral may be corrected.

PROB. XXXIV.

To find (s) the sum of $v-1$ terms of the series, $\frac{a}{1 \cdot d+1} + \frac{a^2}{2 \cdot d+2}$
 $+ \frac{a^3}{3 \cdot d+3} + \frac{a^4}{4 \cdot d+4} \&c.$

Here the v^{th} term or $s = \frac{a^v}{v \cdot d+v} = \frac{a^v}{v \cdot v} = a^v \times \frac{1}{v \cdot v} +$
 $\frac{1-d}{v \cdot v \cdot v} + \frac{1-d \cdot 2-d}{v \cdot v \cdot v} + \frac{1-d \cdot 2-d \cdot 3-d}{v \cdot v \cdot v} \&c.$

Assume $s = \frac{A}{v \cdot v} + \frac{B}{v \cdot v \cdot v} + \frac{C}{v \cdot v \cdot v} + \frac{D}{v \cdot v \cdot v} \&c. \times a^v,$

Then $s = \frac{Aa}{v \cdot v} + \frac{Ba}{v \cdot v \cdot v} + \frac{Ca}{v \cdot v \cdot v} + \frac{Da}{v \cdot v \cdot v} \times a^v,$

That is, by Examp. 12. Prop. XII.

$$s = \frac{Aa}{v \cdot v} + \frac{Ba-2Aa}{v \cdot v \cdot v} + \frac{Ca-3B}{v \cdot v \cdot v} + \frac{Da-4Ca}{v \cdot v \cdot v} \&c. \times a^v.$$

Then $s - s$ or

$$s = \frac{a-1 \cdot A}{v \cdot v} + \frac{a-1 \cdot B-2Aa}{v \cdot v \cdot v} + \frac{a-1 \cdot C-3Ba}{v \cdot v \cdot v} \&c. \times a^v$$

$$= \frac{1}{v \cdot v} - \frac{d-1}{v \cdot v \cdot v} + \frac{d-1 \cdot d-2}{v \cdot v \cdot v} \&c. \times a^v$$

Then equating the coefficients, $a-1 \cdot A=1$, $a-1 \cdot B-2Aa = -$
 $d-1$. &c. Whence $A = \frac{1}{a-1}$, $B = \frac{2Aa-d-1}{a-1}$,

$$C = \frac{3Ba+d-1 \cdot d-2}{a-1}, D = \frac{4Ca-d-1 \cdot d-2 \cdot d-3}{a-1}, \&c.$$

Therefore

$$\text{Therefore } s = \frac{1}{a-1 \cdot v \cdot v} + \frac{2Aa - \overline{d-1}}{a-1 \cdot v \cdot v \cdot v} + \frac{3Ba + \overline{d-1} \cdot \overline{d-2}}{a-1 \cdot v \cdot v \cdot v} +$$

$$\frac{4Ca - \overline{d-1} \cdot \overline{d-2} \cdot \overline{d-3}}{a-1 \cdot v \cdot v \cdot v} + \frac{5Da + \overline{d-1} \cdot \overline{d-2} \cdot \overline{d-3} \cdot \overline{d-4}}{a-1 \cdot v \cdot v \cdot v \cdot v} \&c.$$

Take then $v-1=10$ (or any number of) terms, then $s=10t$ (the sum of ten terms); then we have,

$$10t = a^{11} \times \text{into } \frac{1}{a-1 \cdot 11 \cdot 12} + \frac{2Aa - \overline{d-1}}{a-1 \cdot 11 \cdot 12 \cdot 13} \&c. \text{ whence}$$

by correction

$$s - 10t = a^v \times \frac{1}{a-1 \cdot v \cdot v} \&c. - a^{11} \times \frac{1}{a-1 \cdot 11 \cdot 12} \&c.$$

that is,

$$s = 10t + \frac{a^v}{a-1} : \times \frac{1}{a-1 \cdot v \cdot v} + \frac{2Aa - \overline{d-1}}{a-1 \cdot v \cdot v \cdot v} + \frac{3Ba + \overline{d-1} \cdot \overline{d-2}}{a-1 \cdot v \cdot v \cdot v} \&c.$$

$$- a^{11} : \times \frac{1}{a-1 \cdot 21 \cdot 12} + \frac{2Aa - \overline{d-1}}{a-1 \cdot 11 \cdot 12 \cdot 13} + \frac{3Ba + \overline{d-1} \cdot \overline{d-2}}{a-1 \cdot 11 \cdot 12 \cdot 13 \cdot 14} \&c.$$

Hence if a be less than 1, the sum of all the terms *ad infinitum* =

$$10t + a^{11} \times : \frac{1}{1-a \cdot 11 \cdot 12} + \frac{2Aa - \overline{d-1}}{1-a \cdot 11 \cdot 12 \cdot 13} + \frac{3Ba + \overline{d-1} \cdot \overline{d-2}}{1-a \cdot 11 \cdot 12 \cdot 13 \cdot 14}$$

+ &c. where A, B, C, &c. are the foregoing coefficients, exclusive of v, v, v &c.

P R O B. XXXV.

To find the sum of the powers of the arithmetical progression,

$$a^m + a+e^m + a+2e^m + a+3e^m \&c.$$

To find s the sum of v terms of this series; the last term will be $a+v-1.e^m$, and the next succeeding term or $s = a+ve^m$;

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put

put $a+ve=x$, then $x=ev=e$, and $s=x^m$. And by Cor. Ex. 10.

Prop. XIII. the integral is

$$s = \frac{x^{m+1}}{m+1 \cdot x} - \frac{x^m}{2} + \frac{mx^{m-1}}{12} - \frac{m \cdot m-1 \cdot m-2 \cdot x^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^{m-3} \&c.$$

that is,

$$s = x^{m+1} \times : \frac{1}{m+1 \cdot e} - \frac{1}{2x} + \frac{me}{3 \cdot 4 \cdot x^3} - \frac{m \cdot m-1 \cdot m-2 \cdot e^3}{1 \dots 6 \cdot x^4} +$$

$$\frac{m \dots m-4 \cdot e^5}{1 \dots 7 \cdot 6x^6} - \frac{m \dots m-6 \cdot e^7}{1 \dots 8 \cdot 5 \cdot 6x^8} + \frac{5m \dots m-8 \cdot e^9}{1 \dots 10 \cdot 11 \cdot 6x^{10}} \&c. \text{ con-}$$

tinued till the index of x be 1 or 2, which may be corrected by summing up 10 or more terms, &c. or otherwise when m is an integer.

If $a=e$, then $x=v+1 \cdot a$.

COROLLARY.

The sum of v terms of the series $\frac{1}{a^r} + \frac{1}{(a+e)^r} + \frac{1}{(a+2e)^r} +$
 $\frac{1}{(a+3e)^r} \&c.$ will be $= \frac{1}{x^{r-1}} \times$ into $-\frac{1}{r-1 \cdot e} - \frac{1}{2x} - \frac{re}{3 \cdot 4 \cdot xx}$
 $+ \frac{r \cdot r+1 \cdot r+2 \cdot e^3}{1 \dots 6 \cdot x^4} - \frac{r \dots r+4 \cdot e^5}{1 \dots 7 \cdot 6x^5} \&c.$

which may be corrected by seeking ten or twelve terms of the given series.

PROB. XXXVI.

To find (s) the sum of x terms of the harmonical progression,

$$\frac{1}{a} + \frac{1}{a+e} + \frac{1}{a+2e} + \frac{1}{a+3e} \&c.$$

The last term will be $\frac{1}{a+x-1 \cdot e}$, and the next succeeding term

or $s = \frac{1}{a+xe}$. Put $z=a+xe$, then $z=ex=e$. And $s = \frac{1}{z}$, and
 (by Examp. 12. Prop. XIII.) the integral is

$$s =$$

$$s = \frac{Z}{e} - \frac{1}{2z} - \frac{e}{12zz} - \frac{ee}{12z.z} - \frac{19e^3}{120z.z.z} - \frac{9e^4}{20z.z.z} - \frac{863e^5}{504z.z.z.z} \&c. + Q. \text{ where } Z = \text{hyp. log. } z.$$

Or thus,

$$\text{Since } s = \frac{1}{z}, \text{ the integral is } s = \frac{Z}{e} - \frac{1}{2z} - \frac{e}{12zz} + \frac{e^3}{120z^4} - \frac{e^5}{252z^6} + \frac{e^7}{240z^8} \&c. + Q.$$

E X A M P L E.

The sum of $z-1$ terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \&c.$ is $= Z + .577214 - \frac{1}{2z} - \frac{1}{12zz} - \frac{1}{12z.z.z} - \frac{19}{120z.z.z} \&c.$

or the sum of x terms $= X + .577213 + \frac{1}{2x} - \frac{1}{12xx} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} \&c.$

P R O B. XXXVII.

To find the sum (s) of n terms of the series, $\frac{1.2}{3.4} + \frac{2.3}{4.5} + \frac{3.4}{5.6} + \frac{4.5}{6.7} + \frac{5.6}{7.8} \&c.$

The n^{th} term is $\frac{n.n+1}{n+2.n+3}$, and $s = \frac{n+1.n+2}{n+3.n+4}$; put $v=n+3$,

and $v=n=1$. Then $s = \frac{v-2.v-2}{v.v}$, and by Example 20.

Prop. XIII. the int. $s = \frac{v}{v} - 2r \left[\frac{1}{v} \right] - \frac{r-v}{v.v}r$. Where $r=2, v=1$. Put $v=z, Z = \text{hyp. log. } z$. Then

$$s =$$

$$s = v - \frac{2}{v} - 4 \left[\frac{1}{z} \right] = v - \frac{2}{v} - 4Z + \frac{2}{z} + \frac{1}{3z^2} + \frac{1}{3z^2} + \frac{19}{30z^3} + \frac{9}{5z^4} \text{ \&c.}$$

But when $n=6$, $v=9$, v or $z=10$, $Z=2.302585$, and the sum of 6 terms $= 2.461904$; whence by correction, $s - 2.461904 = v - \frac{2}{v} - 4Z + \frac{2}{z} + \frac{1}{3z^2} \text{ \&c.} = 9 + \frac{2}{9} + 9.210340 - \frac{2}{10}$

$$- \frac{1}{3.10.11} - \frac{1}{3.10.11.12} - \frac{19}{30.10.13} \text{ \&c. that is,}$$

$$s = 2.691136 + v - \frac{2}{v} - 4Z + \frac{2}{z} + \frac{1}{3z^2} + \frac{1}{3z^2} + \frac{19}{30z^3} + \frac{9}{5z^4} + \frac{863}{126z^5} \text{ \&c.}$$

PROB. XXXVIII.

To find n terms of the series, $1 + \frac{3}{1.3} + \frac{3.4}{1.2.3^2} + \frac{4.5}{2.3^3} + \frac{5.6}{2.3^4} + \frac{6.7}{2.3^5} + \frac{7.8}{2.3^6} \text{ \&c.}$

Let s = the sum of n terms, then the n^{th} term $= \frac{n \cdot \overline{n+1}}{2.3^{n-1}}$, and the $\overline{n+1}^{\text{th}}$ term or $s = \frac{\overline{n+1} \cdot \overline{n+2}}{2.3^n}$, let $v = n+1$, then $v = n+1$.

and $s = \frac{v \cdot v}{2.3^n}$. And by Example 14. Prop. XIII. the integral $s =$

$$\frac{v \cdot v}{2.3^n R} - \frac{2}{3.Rv} A - \frac{1}{3.Rv} B, \text{ where } R = -\frac{2}{3}. \text{ That is,}$$

$$s =$$

$$s = -\frac{v \cdot v}{4 \cdot 3^{n-1}} - \frac{v}{4 \cdot 3^{n-1}} - \frac{1}{8 \cdot 3^{n-1}} = \frac{-1}{4 \cdot 3^{n-1}} \times \frac{v \cdot v + \frac{1}{2}}{1 \cdot 1}, \text{ but}$$

when $s=0$, $n=0$, $v=1$, $v=2$, and the integral corrected is, $s = \frac{27}{8}$

$$- \frac{1}{4 \cdot 3^{n-1}} \times \frac{v \cdot v + \frac{1}{2}}{1 \cdot 1}.$$

PROB. XXXIX.

To find the sum of 100 terms of the series, $\frac{2}{1} + \frac{4}{3}a + \frac{6}{5}b + \frac{8}{7}c + \frac{10}{9}d$ &c.

Let t be any term, x its place; then the next term is $\frac{2x+2}{2x+1} t = t$.
 Put $z=2x+1$, then $z=2x=2$. And $\frac{z+1}{z} t = t$, or $\frac{z+1}{z} s = s$. Then
 by Examp. 23. Prop. XIII. $s = \frac{z-z}{z+n} s$, where $n=1$; that is,
 $s = \frac{z-z}{2+1} s = \frac{z}{3} s$; but when $s=0$, $x=1$, $z=3$, $s=2$; therefore by
 correction $s = \frac{z}{3} - \frac{2}{3} = \frac{2x-1}{3} s - \frac{2}{3}$. Here in this case t or s
 is the 101th term, $x=101$, $z=203$. Therefore $s = \frac{201}{3} s - \frac{2}{3} =$
 $67 s - \frac{2}{3}$.

Now to find s by Prob. XIV. $z=102$, $x=203$, then

$$s s = 1.570796 \times : 203 - \frac{1}{2} + \frac{1}{8 \cdot 203} + \frac{1}{16 \cdot 203 \cdot 205} +$$

$$\frac{.086}{203 \cdot 205 \cdot 207} \text{ \&c : } = 318.08716, \text{ and } s = 17.835; \text{ therefore}$$

$$s = 67 \times 17.835 - \frac{2}{3} = 1194.9448 - .6666 = 1194.2782.$$

Hence in general, the sum of $x-1$ terms of the series is $\frac{2x-1}{3} s - \frac{2}{3}$;
 and therefore the sum of this series continued *ad infinitum*, is infinite.

PROB. XL.

To find the sum of 1000 terms of the series, $1 + \frac{1}{4}A + \frac{1}{6}B + \frac{1}{8}C + \frac{1}{10}D + \frac{1}{12}E$ &c.

Let t, t_1 be two terms of the series, succeeding each other, x the place of t , $t = s$, then $t_1 = \frac{2x-3}{2x}t$, or $\frac{x-\frac{3}{2}}{x}t = t_1$, and let s = sum of $x-2$ terms. Then $\frac{x-\frac{3}{2}}{x}s = s_1$. And by Examp. 23. Prop. XIII. $s = \frac{x-x}{n+x}s$, where $n = -\frac{1}{2}$, $x=1$. Therefore $s = \frac{x-1}{1-\frac{1}{2}}s = \frac{x-1}{-\frac{1}{2}}s = -\frac{2x-2}{1}s$. But when $s=0$, $s_1=1$, $x-2=0$, or $x=2$. Therefore by correction $s = 2 - \frac{2x-2}{1}s$.

Now to find s or t , the $\frac{1}{x-1}$ term.

By Prob. XVI. we have $t = Ax^{-\frac{3}{2}} + Bx^{-\frac{5}{2}} + Cx^{-\frac{7}{2}} + Dx^{-\frac{9}{2}}$ &c.

Where

$$B = \frac{-\frac{3}{2} \times -\frac{5}{2}}{2} A = \frac{3 \cdot 5}{8} A = \frac{15}{8} A.$$

$$C = -\frac{3 \cdot 5 \cdot 7}{3 \cdot 3^2} A + \frac{5 \cdot 7 \cdot 15}{16 \cdot 8} A = \frac{11 \cdot 5 \cdot 7}{4 \cdot 3^2} A = 3.0078125 A.$$

$$D = \frac{3 \cdot 5 \cdot 7 \cdot 9}{2^5 \cdot 3 \cdot 3 \cdot 4} A - \frac{5 \cdot 7 \cdot 9 \cdot 15}{2^4 \cdot 3 \cdot 3 \cdot 8} A + \frac{3 \cdot 7}{2^3} \times 3.0078 A = 4.61426 A.$$

$$E = 6.9720 A.$$

$$F = 10.5195 A.$$

But $t = \frac{1}{\sqrt{x}} \times : \frac{A}{x} + \frac{B}{xx} + \frac{C}{x^3} + \frac{D}{x^4}$ &c. that is,

$$t \text{ or } s = \frac{A}{\sqrt{x}} \times : \frac{1}{x} + \frac{1.875}{xx} + \frac{3.007812}{x^3} + \frac{4.61426}{x^4} + \frac{6.9720}{x^5} + \frac{10.519}{x^6} \text{ &c.}$$

Now

Now let t be the 9th term, $x-1=9$, or $x=10$; then

$$t = \frac{1.1.3.5.7.9.11.13.15}{4.6.8.10.12.14.16.18} = \frac{5.11.13}{2.4^7} = .021820.$$

Whence

$$.021820 = \frac{A}{\sqrt{10}} \times \frac{1}{10} + \frac{1.875}{100} + \frac{3.00781}{1000} + \frac{4.61426}{10000} \\ + \frac{6.972}{100000} + \frac{10.5}{1000000} \&c.$$

And collecting the terms,

$$\begin{array}{r} .100000 \\ 18750 \\ 3008 \\ 461 \\ 69 \\ 11 \\ 2 \\ \hline .122301 \end{array}$$

$$.021820 \sqrt{10} = A \times .122301, \text{ whence } A = .564188.$$

$$\text{Therefore } s = 2 - \frac{2x-2}{1} \times \frac{.564188}{\sqrt{x}} \times \frac{1}{x} + \frac{1.875}{x x} + \\ \frac{3.0078}{x^3} + \frac{4.614}{x^4} + \frac{6.97}{x^5} + \frac{10.5}{x^6} \&c.$$

Now let $x-2=1000$, or $x=1002$, then

$$s = 2 - \frac{2002 \times .564188}{\sqrt{1002}} \times \frac{1}{1002} + \frac{1.875}{1004004} \&c. = 2 - \\ 35.682404 \times .009982 = 2 - .356182, \text{ that is,}$$

$$s = 1.643818, \text{ the sum of 1000 terms.}$$

Hence if x be infinite, the sum of all the terms *ad infinitum* is =2.

PROB. XLI.

To find (s) the sum of z terms of the series, $\frac{1.2}{3.4.4} + \frac{2.3}{4.5.4^2} +$
 $\frac{3.4}{5.6.4^3} + \frac{4.5}{6.7.4^4} + \frac{5.6}{7.8.4^5} \&c.$

Here the z^{th} term $= \frac{z \cdot \overline{z+1}}{z+2 \cdot \overline{z+3} \cdot 4^z}$, and the $\overline{z+1}^{th}$ term or

$$s = \frac{\overline{z+1} \cdot \overline{z+2}}{z+3 \cdot \overline{z+4}} \times \frac{1}{4} \Big|^{z+1} \cdot \text{Put } v=z+3, \text{ then } v=z+4, v=z+1.$$

$$\text{Then } s = \frac{\overline{v-2} \cdot \overline{v-1}}{v \cdot v} \times \frac{1}{4} \Big|^{z+1} \cdot \text{Compare this increment with Ex-}$$

ample 22. Prop. XIII. and we have $r=2$, $a=\frac{1}{4}$, $R = \frac{1}{4} - 1 = -\frac{3}{4}$,
 $z=z+1$, $v=v$, whence by that Example we have

$$s = -\frac{1}{3 \cdot 4^z} \times \text{into } 1 - \frac{4}{v} - \frac{1}{3^v} A \left. \begin{array}{l} - \frac{2}{3^v} B - \frac{3}{3^v} C \\ + \frac{6}{v \cdot v} \end{array} \right\} - \frac{4}{3^v} D - \frac{5}{3^v} E - \frac{6}{3^v} F \&c.$$

Now let $z=6$, then $v=9$, and the sum of 6 terms, computed from
the given series, will be .069191; therefore by correction,

$$\begin{aligned} .069191 &= Q - \frac{1}{12288} \times \text{into } 1 - \frac{4}{9} - \frac{1}{3^9} A \left. \begin{array}{l} + \frac{6}{9 \cdot 10} \end{array} \right\} - \frac{2}{3 \cdot 11} B \\ &- \frac{3}{3 \cdot 12} C - \frac{4}{3 \cdot 13} D \&c. = Q - \frac{1}{12288} \times \text{into } .632473 = Q \\ &- .000052, \text{ whence } Q = .069243. \text{ And therefore} \end{aligned}$$

$s =$

$$s = .069243 - \frac{1}{3 \cdot 4^2} \times \text{into } 1 - \frac{4}{v} - \frac{1}{3^v} A \left. \vphantom{\frac{1}{3^v} A} \right\} - \frac{2}{3^v} B - \frac{3}{3^v} C \text{ \&c.}$$

$$+ \frac{6}{v^v} \left. \vphantom{\frac{6}{v^v}} \right\}$$

It is evident the sum of the whole series *ad infinitum* is $= .069243$.

PROB. XLII.

To find s the sum of $z-1$ terms of the series, $1 + \frac{2 \cdot 4}{3 \cdot 3} A + \frac{4 \cdot 6}{5 \cdot 5} B$
 $+ \frac{6 \cdot 8}{7 \cdot 7} C + \frac{8 \cdot 10}{9 \cdot 9} D \text{ \&c.}$

Let the z^{th} term be v , then $\frac{2z \times 2z+2}{2z+1} v = v$;

But by the process in Prob. XVII. we have $v = A \times : 1 + \frac{1}{2x}$
 $+ \frac{5}{8xx} + \frac{1 \cdot 9375}{xxx} \text{ \&c.}$ where $x = 2z+1$; and $A = .78539816$;
 that is,

$$s = A \times : 1 + \frac{1}{2x} + \frac{5}{8xx} + \frac{1 \cdot 93}{xxx} \text{ \&c. and finding the integral,}$$

$$s = Q + A \times : \frac{x}{x} + \frac{1}{2} \left[\frac{1}{x} \right] - \frac{5}{8xx} - \frac{1 \cdot 937}{2xxx} - \frac{9 \cdot 76}{3x \dots x} -$$

$$\text{\&c. But by Examp. 12. Prop. XIII. } \left[\frac{1}{x} \right] = \frac{X}{x} - \frac{1}{2x} -$$

$$\frac{x}{12xx} - \frac{x^2}{12x \dots x} - \frac{19x^3}{120x \dots x} - \frac{9x^4}{20x \dots x} \text{ \&c. Therefore}$$

$$s = Q + A \times : \frac{x}{x} + \frac{X}{2x} - \frac{1}{4x} - \frac{1}{12xx} - \frac{2}{12x \dots x} - \frac{19 \times 4}{120x \dots x} \text{ \&c.}$$

$$- \frac{5}{16x} - \frac{1 \cdot 9375}{4xx} - \frac{9 \cdot 76}{6x \dots x} - \frac{68 \cdot 45}{8x \dots x} \text{ \&c.}$$

where $X = \text{hyp. log. } x$, and $x = 2$.

N

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That is,

$$s = Q + A \times : \frac{x}{2} + \frac{X}{4} - \frac{9}{16x} - \frac{6,8125}{12xx} - \frac{10,76}{6x \dots x} \\ - \frac{9,189}{x \dots x} - \frac{65,2}{x \dots x} \&c.$$

the sum of $z-1$ terms.

Now let $z-1=10$, or $z=11$, then $x=23$, $x=2$; and then

$$s = Q + A \times : \frac{23}{2} + \frac{3,135494}{4} - \frac{9}{16.23} - \frac{6,8125}{12.23.25} - \\ \frac{10,76}{6.23.25.27} - \frac{9,189}{23 \dots 29} - \frac{65,2}{23 \dots 31} \&c.$$

Then collecting the several terms, and summing up 10 terms of the given series, which is 8.459536, we have

$$8.459536 = Q + A \times 12.258290 \\ = Q + 9.627638,$$

Whence

$$Q = -1.168102;$$

Therefore

$$s = -1.168102 + A \times : \frac{x}{2} + \frac{X}{4} - \frac{9}{16x} - \frac{6.8125}{12xx} \\ - \frac{10.76}{6x \dots x} \&c.$$

And hence, the sum of all the terms *ad infinitum*, is infinite.

$$\begin{array}{r} + 11.500000 \\ + .783873 \\ - .024457 \\ - .987 \\ - .115 \\ - .20 \\ - .4 \\ \hline + 12.283873 \\ - .025583 \\ \hline + 12.258290 \end{array}$$

PROB.

PROB. XLIII.

To find the sum of x terms of the series, $1 + \frac{9}{8}A + \frac{25}{24}B$
 $+ \frac{49}{48}C + \frac{81}{80}D + \frac{121}{120}E \text{ \&c.}$

First, find the $\overline{x+1}^{th}$ term by Prob. XVIII. Let the $\overline{x+1}^{th}$
 term be t ; then by the series, $t = \frac{\overline{2x+3}^2}{\overline{2x+3}^2 - 1} t$; put $z = 2x+3$,

then $\frac{z}{z-1} t = t$, here $r = -1$, $z = 2x+3$. Then

$$t = A - \frac{1}{2z} P + \frac{3}{4z} Q + \frac{15}{6z} R + \frac{35}{8z} S \text{ \&c.}$$

Let $x+1=10$, or $x=9$, then $z=21$.

Then calculate the 10th term of the given series, which is 1.241816;
 then we shall have

$$1.241816 = A \times 1 - \frac{1}{42} P + \frac{3}{4 \cdot 23} Q + \frac{15}{6 \cdot 25} R + \frac{35}{8 \cdot 27} S \text{ \&c.}$$

$$= A \times .975320$$

Whence $A = 1.273240$.

Then to find the sum of the series, we have the $\overline{x+1}^{th}$ term or
 $t = 1.273240 \times 1 - \frac{1}{2z} P + \frac{3}{4z} Q + \frac{15}{6z} R \text{ \&c. that is,}$

$$s = A \times : 1 - \frac{1}{2z} - \frac{3}{2 \cdot 4 z z} - \frac{3 \cdot 15}{2 \cdot 4 \cdot 6 z z z} - \frac{3 \cdot 15 \cdot 35}{2 \cdot 4 \cdot 6 \cdot 8 z \dots z} \text{ \&c.}$$

Then finding the integral,

$$s = Q + A \times : \frac{z}{z} - \frac{1}{2} \left[\frac{1}{z} \right] + \frac{3}{8 z z} + \frac{3 \cdot 15}{4 \cdot 2 \cdot 4 \cdot 6 z z} +$$

$$\frac{3 \cdot 15 \cdot 35}{6 \cdot 2 \cdot 4 \cdot 6 \cdot 8 z \dots z} + \frac{3 \cdot 15 \cdot 35 \cdot 63}{8 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 z \dots z} \text{ \&c.}$$

And

And since $\left[\frac{1}{z}\right] = \frac{Z}{z} - \frac{1}{2z} - \frac{z}{12zz_1} - \frac{z^2}{12z \dots z_2} -$

$\frac{19z^3}{120z \dots z_3} \&c.$ where $Z = h. \log. z.$

Therefore finding the numeral coefficients of all the terms, we shall have

$$s = Q + A \times: \frac{z}{2} + \frac{.1875}{z} + \frac{.234375}{zz_1} + \frac{.6836}{z \dots z_2} + \frac{3.2300}{z \dots z_3} + \&c.$$

$$- \frac{Z}{4} + \frac{.25}{z} + \frac{.08333}{zz_1} + \frac{.1666}{z \dots z_2} + \frac{.6333}{z \dots z_3} + \&c.$$

That is,

$$s = Q + A \times: \frac{2z-Z}{4} + \frac{.4375}{z} + \frac{.3177}{zz_1} + \frac{.8502}{z \dots z_2} + \frac{3.8633}{z \dots z_3}$$

$$+ \frac{24.920}{z \dots z_4} + \frac{210}{z \dots z_5} + \frac{2180}{z \dots z_6} \&c.$$

Now to find Q , for correcting the integral; take $x =$ any number, as suppose 10, then $z=23$; then sum up 10 terms of the given series, which is 11.868631; then

$$11.868631 = Q + A \times: \frac{46-3.135494}{4} + \frac{.4375}{23} + \frac{.3177}{23.25} +$$

$$\frac{.8502}{23.25.27} \&c. = Q + A \times 10.735766 = Q + 13.669205; \text{ then}$$

$$Q = -1.800574;$$

Therefore in general

$$s = -1.800574 + A \times: \frac{2z-Z}{4} + \frac{.4375}{z} + \frac{.3177}{zz_1}$$

$$+ \frac{.8502}{zzz_{12}} \&c.$$

PROB. XLIV.

To find (s) the sum of $x-1$ terms of the series, $1 + \frac{1.1}{2.3} A + \frac{3.3}{4.5} B$
 $+ \frac{5.5}{6.7} C + \frac{7.7}{8.9} D$ &c. expressing $\frac{1}{4}$ the circumference of the
 circle whose radius is 1.

Let x be the place of any term t , then $t = \frac{2x-1}{2x \cdot 2x+1} t$, or

$$s = \frac{2x-1}{2x \cdot 2x+1} s; \text{ put } z=2x, \text{ then } z=2x=2, \text{ and } \frac{z-1}{z \cdot z+1} s$$

$$= s = \frac{z^2-2z+1}{z^2+z} s. \text{ Compare this with Examp. 28. Prop. XIII.}$$

and we have $m=-2$, $n=1$, $p=1$, $q=-3$. From hence we shall find

$$A = -1, B = \frac{2}{3}, C = -\frac{4}{15}, D = -\frac{12}{35}, E = -\frac{20}{63},$$

$$F = -\frac{140}{33}, G = -26.43 \text{ \&c.}$$

Therefore

$$s = s \times : -z + \frac{2}{3} - \frac{4}{15z} - \frac{12}{35zz} - \frac{20}{21z \cdot z} - \frac{140}{33z \cdot z} \text{ \&c.}$$

Sum up 10 terms of the given series, which is 1.39169463, then the
 11th term = .008390337, put these for s and s , then $x-1=10$, or
 $x=11$, and $z=22$, then we shall have, by correction,

$$s - 1.3916946 = s \times : -z + \frac{2}{3} - \frac{4}{15z} - \frac{12}{35zz} \text{ \&c.}$$

$$- .00839034 \times : -22 + \frac{2}{3} - \frac{4}{15 \cdot 22} - \frac{12}{35 \cdot 22 \cdot 24} - \frac{20}{21 \cdot 22 \cdot 26}$$

$$- \frac{140}{33 \cdot 22 \cdot 28} \text{ \&c. and summing up this last series,}$$

$$s = 1.39169463 + .00839034 \times 21.3461869 +$$

$$s \times : -z + \frac{2}{3} - \frac{4}{15z} \text{ \&c.} = 1.3916946 + .1791017$$

$$+ s \times : -z + \frac{2}{3} \text{ \&c.}$$

O o

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That is,

$$s = 1.5707963 - s \times : z - \frac{2}{3} + \frac{4}{15z} + \frac{12}{35z^2} + \frac{20}{21z \dots z} + \frac{140}{33z \dots z} + \frac{26.43}{z \dots z} \text{ \&c. the sum of } x-1 \text{ terms of the series.}$$

Now to find s , any required term of the series: This may be done by Prob. XIX, or easier by Prob. XXI. It is plain, the successive terms of the given series, expressed at length, will be

$$1, \frac{1.1.1}{2.3}, \frac{1.1.1.3.3}{2.3.4.5}, \frac{1.1.1.3.3.5.5}{2.3.4.5.6.7}, \text{ \&c. that is, } 1, \frac{1}{2.3}, \frac{1.3}{2.4.5}, \frac{1.3.5}{2.4.6.7}, \text{ \&c. therefore the } x^{\text{th}} \text{ term or } s \text{ will be}$$

$$\frac{1.3.5.7 \dots 2x-3}{2.4.6.8 \dots 2x-2} = \frac{1.3.5 \dots 2x-3}{2.4.6 \dots 2x-2} \times \frac{1}{2x-1}.$$

Therefore if we find the log. of $\frac{1.3.5 \dots 2x-3}{2.4.6 \dots 2x-2}$, and divide the number answering thereto, by $2x-1$, we shall have the value of s , or the x^{th} term of the series. Therefore comparing this with Prob. XXI, we have $a=1$, $b=2$, $r=2$, $n=-1$, $x=2$, $y=2x-2$, $z=(y+r)=2x=z$. Whence $A=-\frac{1}{2}$, $B=\frac{3}{4}$, $C=\frac{1}{2}$, $D=\frac{3}{8}$, $E=\frac{1}{4}$, $F=\frac{1}{16}$, \&c. and $M = .43429448$;

Hence we shall have s or the log. of the x^{th} term $= -\frac{1}{2}Z + M \times : \frac{3}{4z} + \frac{1}{2zz} + \frac{3}{8z^3} + \frac{1}{4z^4} + \frac{3}{20z^5} \text{ \&c. But when } x=11,$
 $z=22.$ And the 11th term being $.0083903 = \frac{1.3 \dots 19}{2.4 \dots 20.21}$

Therefore multiply by 21, and we have $\frac{1.3 \dots 19}{2.4 \dots 20} = .1761963$, and its log. $= -1.2459968 = P$; but when $x=22$, $z=22$, $S=P$; therefore by correction,

$$S-P = -\frac{1}{2}Z + M \times : \frac{3}{4z} + \frac{1}{2zz} + \frac{3}{8z^3} + \text{\&c.} + \frac{1}{2}1.:22 -$$

$$M \times : \frac{3}{4.22} + \frac{1}{2.22^2} + \frac{3}{8.22^3} + \frac{1}{4.22^4} + \frac{3}{20.22^5} \text{ \&c. the last}$$

series

series summed up, and the equation reduced, $S = -.0980617 - \frac{1}{2}Z + M \times : \frac{3}{4z} + \frac{1}{2zz} + \frac{3}{8z^3} + \frac{1}{4z^4} + \frac{3}{20z^5} \&c.$

Now suppose 1000 terms of the series be required, then $x=1001$, and $z=2002$; and therefore putting for z and Z their values, and collecting the terms of the series,

$$S = -.0980617 - 1.6505693 = - 1.7486310;$$

Or rather $- 2.2513690$ for the logarithm S :

Then if the number answering thereto be divided by 2001 ($2z-1$), you have the term required; or (which is the same thing),

From the log. $- 2.2513690$

Subtract the log. 2001 3.3012471

Remains the log. $- 6.9501219$

Whose number is .000008915012

being the term s required.

Laftly,

$$s = 1.5707963 - s \times : z - \frac{2}{3} + \frac{4}{15z} + \frac{12}{35zz} \&c.$$

$$= 1.5707963 - .000008915012 \times 2001.333466$$

$$= 1.5707963 - .0178419 = 1.5529544, \text{ the sum of 1000}$$

terms of the series, as required.

The sum of all the series, *ad infinitum*, is 1.5707963;

For when x and z are indefinitely great, then $s \times : z - \frac{z}{3} \&c.$ va-

nishes; for then it becomes $\frac{-\frac{1}{2}Z}{z-1} \times z$, or $-\frac{1}{2}Z$, and therefore the greater z is, the leffer the fraction will be whose log. is the negative number $-\frac{1}{2}Z$.

PROB. XLV.

To find (s) the sum of 50 terms of the recurrent series, $\overset{(A)}{1} + \overset{(B)}{3x}$
 $+ \frac{\overset{(C)}{3xB - xxA}}{1} + \frac{\overset{(D)}{3xC - xxB}}{1} + \frac{\overset{(E)}{3xD - xxC}}{1} \&c.$ that is,
 $1 + 3x + 8x^2 + 21x^3, \&c.$ where $x = \frac{1}{2}$.

Here s , s_1 , and s_2 are the three next succeeding terms, and by the nature of the series, $s = 3s_1x - s_2x^2$, or $s_2x^2 - 3s_1x + s = 0$; compare this with Examp. 29. Prop. XIII. and we have $m = xx$, $n = -3x$, $p = 1$, $q, \&c. = 0$. Therefore we have

$$s = -\frac{-3x + 1 \times s_1 + s_2}{xx - 3x + 1} = \frac{3x - 1 \cdot s_1 - s_2}{xx - 3x + 1}, \text{ but when } s=0, s_1=1, s_2=3x, \text{ and the integral corrected is}$$

$$s = \frac{3x - 1 \cdot s_1 - s_2 + 1}{xx - 3x + 1}.$$

Now all the difficulty will be to find s and s_1 ; for this end continue the series forward; which is done, by taking three times the coefficient of any term, and from that subtracting the coefficient of the foregoing; for the coefficient of the next following term, as appears by the general construction of the series; and as the power of x is known for any term, this difficulty lies in finding the coefficients; the series continued out is,

$$1 + 3x + 8xx + 21x^3 + 55x^4 + 144x^5 + 377x^6 + 987x^7 + 2584x^8 \\ + 6765x^9 + 17711x^{10} + 46368x^{11} \&c.$$

But as the producing all the terms as far as the term required, is a great deal of labour, we shall come at it shorter by the following method.

Suppose any series of the coefficients to be M, N, O, P, Q, R, S, T, U, W, X, Y, Z. And put $f=3$, $g=1$, then the scale of proportion, will be $f-g$; and since by construction of the given series,
 $Z =$

$Z = fY - gX$, $Y = fX - gW$, $X = fW - gV$, $W = fV - gT$, &c.
 Therefore by expunging successively, Y , X , W , V , &c. we shall get
 several values of Z as follows:

$$\begin{array}{l}
 Z \\
 fY - gX \\
 \overline{ff - g} \cdot X - fgW \\
 \overline{f^3 - 2fg} \cdot W - \overline{ffg - gg} \cdot V \\
 \text{\&c.}
 \end{array}$$

In numbers:

$$\begin{array}{l}
 Z \\
 3Y - 1X \\
 8X - 3W \\
 21W - 8V \\
 55V - 21T \\
 144T - 55S \\
 377S - 144R \\
 987R - 377Q \\
 2584Q - 987P \\
 6765P - 2584O \\
 \text{\&c.}
 \end{array}$$

Hence we have $6765P - 2584O = Z$. Therefore if O be the n^{th}
 term, from the beginning of the series; and Z the 10th term from
 O ; then will $Z = \overline{n + 10}^{\text{th}}$ term of the series. By this means we
 can pass from the 10th term to the 20th, from the 20th to the 30th,
 and so on.

But further; since the numbers computed backward from Z , are
 formed the same way as the several terms of the given series; it is
 plain, these numbers can be no other than the terms of the said
 series. They are found the same way, for in any term, the left-hand
 number is multiplied by f , and the right by g , and subtracted, gives
 the next left-hand number. The right-hand numbers are had by mul-
 tiplying the foregoing left-hand numbers by g . These numbers then
 being the terms of the given series; let a , b be the 9th and 10th
 terms of the series, then $bP - aO = \overline{n + 10}^{\text{th}}$ term, as shewn before;
 and for the same reason if a , b are the 10th and 11th terms,

P p

bP - aO

$bP - aO = \overline{n+1}^{tb}$ term, and so on. Whence in general, if O, P be the n^{th} and $\overline{n+1}^{tb}$ terms; and a, b the m^{th} and $\overline{m+1}^{tb}$ terms then will $bP - gaO = \overline{n+m+1}^{tb}$ term; or in our case $bP - aO = \overline{n+m+1}^{tb}$ term. By this rule one may proceed from the 20th and 30th terms to the 50th, and so the 100th, the 200th, the 400th, the 800th, the 1000th term, &c.

There are other methods of doing it, but this method of approach is sufficient for our purpose.

In the present case,

Let $O = 6765$ (the 10th term) $P = 17711$ (the 11th term), and there will be had

$$Z = 102334155 \text{ the 20th term.}$$

If $O = 17711, P = 46386$, then

$$Z = 267914296 \text{ the 21st term.}$$

From the 20th and 21st terms, there will be found by the progression of the given series,

$$701408733 = 22d \text{ term.}$$

Again, putting a, b the 9th and 10th terms, and O, P the 20th and 21st, and then the 21st and 22d; the value of $bP - aO$ will give the 30th term $= 1548008(0^6)$, the number 1548008 with 6 cyphers or figures annexed, that is,

$$1548008(0^6) = 30th \text{ term.}$$

$$\text{Likewise } 4052739(0^6) = 31st \text{ term.}$$

$$\text{And thence } 10610210(0^6) = 32d \text{ term.}$$

In like manner,

If a, b be the 20th and 21st terms; and O, P the 30th and 31st, and then the 31st and 32d terms;

$$\text{we shall find } 92737275(0^{11}) = 51st \text{ term.}$$

$$\text{and } 242789338(0^{11}) = 52d \text{ term.}$$

Lastly,

Laftly,

The correspondent powers of x will be had by the help of logarithms ;

Since $x = \frac{1}{2}$, find the refpective powers of 2 ; thus

$$2^{50} = 112590(0^{10}), \text{ and } 2^{51} = 225180(0^{10}).$$

Therefore

$$s = \frac{9273 \text{ \&c.}}{11259 \text{ \&c.}} = 823672.4 ; \text{ and}$$

$$s = \frac{2427 \text{ \&c.}}{225 \text{ \&c.}} = 1078201.2 ;$$

Therefore

$$s = \frac{3x-1 \cdot s - s+1}{xx-3x+1} = \frac{1s-s+1}{-\frac{1}{4}} = 2665456,$$

the fum of 50 terms of the ferief as required.

Many more Examples of folving Problems, might here be given ; but the method is obvious to any intelligent reader.

F I N I S.

INSTRUCTIONS

The corresponding powers of a will be had by the help of logs.

Since $a^2 = 112.90$, and the respective powers of a , thus
 $a^4 = 127.40$, and $a^8 = 141.80$.

Therefore

$$\begin{aligned} \frac{1}{a} &= \frac{1}{112.90} = 8.857 \times 10^{-3} \text{ and } \frac{1}{a^2} = 8.236 \times 10^{-4} \\ \frac{1}{a^3} &= \frac{1}{127.40} = 7.849 \times 10^{-4} \end{aligned}$$

Therefore

$$\frac{1}{a^4} = \frac{1}{141.80} = 7.052 \times 10^{-4}$$

the sum of 50 terms is the times as required.

Many more Examples of this kind might here be given; but the method is obvious to any intelligent reader.



FINIS

